

Open Topological Strings and Integrable Hierarchies: Remodeling the A-Model

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Abstract: We set up, purely in A-model terms, a novel formalism for the global solution of the open and closed topological A-model on toric Calabi-Yau threefolds. The starting point is to build on recent progress in the mathematical theory of open Gromov-Witten invariants of orbifolds; we interpret the localization formulae as relating D-brane amplitudes to closed string amplitudes perturbed with twisted masses through an analogue of the “loop insertion operator” of matrix models. We first generalize this form of open/closed string duality to general toric backgrounds in all chambers of the stringy Kähler moduli space; secondly, we display a neat connection of the (gauged) closed string side to tau functions of 1+1 Hamiltonian integrable hierarchies, and exploit it to provide an effective computation of open string amplitudes. In doing so, we also provide a systematic treatment of the change of flat open moduli induced by a phase transition in the closed moduli space. We test our proposal in detail by providing an extensive number of checks. We also use our formalism to give a localization-based derivation of the Hori-Vafa spectral curves as coming from a resummation of A-model disc instantons.

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1. Introduction

The topological phase of string theory has been a major source of insights both in Mathematics and Physics. Topological strings on Calabi-Yau manifolds come in two guises, which are related by mirror symmetry: the A-model and the B-model. On the Physics side, they yield a great deal of non-trivial information about the vacuum structure of type IIA and IIB superstring compactifications and the holomorphic effective dynamics of the resulting supersymmetric gauge and gravity theories; at the same time, they provide a privileged laboratory for studying general ideas about dualities in string theory, such as mirror symmetry and gauge/string duality. From a mathematical point of view, the A-twisted topological string captures a sophisticated set of invariants of the target manifold in the form of a “virtual” count of holomorphic maps in the form of Gromov-Witten invariants; its B-model mirror symmetric counterpart can instead be

regarded as a quantized version of the theory of variation of Hodge structures on the mirror Calabi-Yau.

A special case, and one that has been subject to intense study in the last decade, is given by the topological A-model on *toric* Calabi-Yau threefolds in the presence of Lagrangian D-branes. From a technical point of view, this setup enjoys a host of desirable features: it provides a local model for the topological string on compact Calabi-Yau threefolds, and at the same time it shares many qualitative features - like the existence of matrix model duals [4, 38, 69], a mirror description in terms of Riemann surfaces [53, 55], and some form of underlying integrability [2, 39, 41] - with the case of topological strings on Fano manifolds.

Essentially two, somewhat complementary formalisms have been put forward to solve the open and closed topological A-model on these backgrounds: the topological vertex formalism of [2] and the “Remodeling the B-model” proposal of [13]. In both cases, the key principle at the base is some string duality: for the topological vertex, topological gauge/string duality with Chern-Simons theory [49] allows to solve the theory to all orders in g_s around large radius; for the remodeled-B-model, a local mirror symmetry picture in terms of dual spectral curves is the starting point for a recursive solution based on the Eynard-Orantin formalism for matrix models [13, 41, 70].

In this paper we develop a formalism to solve the open topological A-model on a toric CY3 target X with toric Lagrangian branes $L \hookrightarrow X$ from a direct A-model instanton analysis, without appealing to string duality. In our setup, genus g , h -holed open string amplitudes are computed from (equivariant) closed string amplitudes via an analogue of the loop insertion operator of matrix models:

$$F_{g,h}^{X,L}(t_1, \dots, t_n; w_1, \dots, w_h; f) = \left(\prod_{i=1}^h \mathcal{L}^X(w_i, f) \right) \mathcal{F}_g^X(t_{\alpha,p}, f) \Big|_{t_{\alpha,p} = \delta_{\deg \phi_\alpha, 2\delta_{p,0} t_\alpha}. \quad (1.1)$$

In the right hand side, $\mathcal{F}_g^X(t_{\alpha,p})$ is the full-descendant genus g free energy of the topological A-model, twisted by a $T \simeq \mathbb{C}^*$ action specified by the location of the brane and the choice of an integer f which captures the framing ambiguity on the dual Chern-Simons side. The chemical potentials $t_{\alpha,p}$ count the insertions of the p^{th} gravitational descendant of a chiral operator $\phi_\alpha \in QH_T^\bullet(X)$, whereas the operator $\mathcal{L}^X(w, f)$ is a first order differential operator in $t_{\alpha,p}$.

Equation (1.1) is the outcome of the localization approach to define and compute open Gromov-Witten invariants on toric Calabi-Yau manifolds first put forward in [59]; in the case of orbifolds of \mathbb{C}^3 , Atiyah-Bott localization [17] results in an explicit expression for the operator $\mathcal{L}(w, f)$. We will build on this along two main directions. We first of all generalize (1.1) to any toric Calabi-Yau threefold, in any patch of the closed string moduli space, also away from, and possibly in the absence of, orbifold points. Secondly, we exhibit a direct connection of the closed, T -equivariant theory with descendants in the r.h.s. of (1.1) with the theory of classical Hamiltonian integrable systems in 1+1 dimensions.

The resulting formalism, which is perturbative in g_s but which holds true globally in α' , can be regarded as an A-model mirror of the remodeled-B-model of [13], and is interesting for a number of reasons. First of all, from a conceptual point of view, it gives a purely A-model formulation of the problem of computing $F_{g,h}^{X,L}$, which coincides with their rigorous (albeit purely calculational) mathematical definition from localization. Second, it is valid for general toric Calabi-Yau threefolds, and in all chambers of the

extended Kähler moduli space of X , including orbifold points. Thirdly, while not being as computationally straightforward as the BKMP-formalism [13], it is still surprisingly effective for computing the type of amplitudes considered in [13], especially considering the fact that its starting point is completely rooted on the A-side, and in some examples it goes beyond the methods known to date. Fourthly, it exhibits a novel, clear connection to underlying integrable structures of the topological string; the resulting picture is quite different from (and in a way simpler than) the one arising from the dual Kodaira-Spencer theory on the B-model side [2], and it provides moreover a clear identification of a key object in the relation of topological strings to matrix models, namely, the brane insertion operator $\mathcal{L}(w, f)$. Fifthly, it embeds in a systematic fashion the change of the canonical choice of flat open string moduli when moving from one chamber to another of the Kähler moduli space. Finally, it can be used to make contact with the results of local mirror symmetry, and most notably to recover the mirror Calabi-Yau geometry based on spectral curves from a resummation of A-model instantons.

The paper is organized as follows. We first review in Sect. 2 the necessary background on the open and closed topological A-model on toric Calabi-Yau threefolds; in view of the role of the master formula (1.1), we discuss in some detail the closed equivariant side and the “recoupling” to topological gravity induced by gauging the T -action. We then describe our formalism in Sect. 3: we first review the derivation of (1.1) for orbifolds and formulate its extension to general toric Calabi-Yau threefolds. We then describe the relation of the closed equivariant model with integrable hierarchies, and discuss its general concrete implementation at low genera. A crucial role here is played by Dubrovin’s theory [30, 31] of dispersionless hierarchies arising from associativity equations, and their dispersive deformation [32] via the group of rational Miura transformations. Section 4 is devoted to our three main examples: the framed vertex, for which we find a relation to a disguised form of the KdV hierarchy, the resolved conifold, where the relevant integrable system is the Toeplitz reduction of the 2D-Toda hierarchy [15, 16], and local \mathbb{P}^2 , where we explicitly test that our formalism correctly computes topological amplitudes possessing a non-trivial quasi-modular dependence on the closed string moduli; we also briefly report on the case of a particular \mathbb{Z}_7 orbifold of \mathbb{C}^3 , for which computations of open string amplitudes at the orbifold point would be awkward (if not unfeasible) with other methods. In Sect. 5 we make contact with toric mirror symmetry and derive in each of our examples the Hori-Vafa spectral curves by summing over open string instantons at $g = 0$, $h = 1$. We conclude in Sect. 6 with some remarks on new possible developments. Some background material on I -functions of toric orbifolds are included in the Appendix.

2. The Open and Closed A-Model on Toric Calabi-Yau Threefolds

2.1. The open string side.

2.1.1. Geometry. We will be interested in the topological A-model on a toric Calabi-Yau threefold (TCY3) X with a background Lagrangian toric brane $L \hookrightarrow X$; we briefly review in this section the geometric setup. There is no new material here; further details may be found in [5, 7, 13, 54, 57, 59].

By definition, a smooth TCY3 X is a Kähler manifold with vanishing canonical class and admitting a complex rank three group of holomorphic isometries, whose (algebraic) maximal torus we denote by $S \simeq (\mathbb{C}^*)^3$. This last fact allows to describe X in a purely diagrammatic way in terms of a three-dimensional integer sublattice of \mathbb{Z}^3 - the *fan*

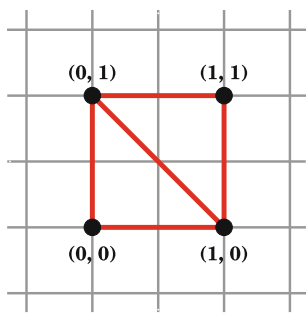


Fig. 1. The toric diagram Σ_X of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

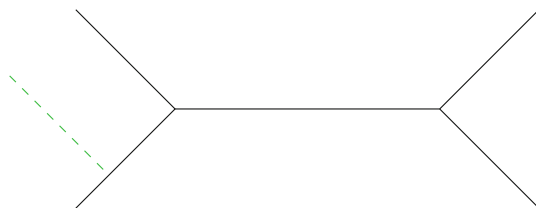


Fig. 2. The web diagram of $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, with a brane on an external leg

\mathcal{F}_X of X - which specifies the way the S -orbits close (see [26, 54] for details). In the Calabi-Yau case which is of our interest, this information can be compactly encoded in a triangulated polytope $\Sigma_X \subset \mathbb{Z}^2$ - the *toric diagram* of X , as in Fig. 1. Alternatively [3], and equivalently, X can be realized as a degenerate $\mathbb{T}^2 \times \mathbb{R}_+$ fibration over \mathbb{R}^3 ; the geometry is completely specified by a finite number of integer data which specify the degeneration loci of the torus fibers. The degeneration locus, which is the image of the moment map by which X is realized, has an easy interpretation in terms of the Newton polytope Σ_X : it is the dual polytope of Σ_X , and goes under the name of the *web diagram* of X . See Fig. 2.

We now want to turn on an open string sector. In the toric setting there is a distinguished set of Special Lagrangian submanifolds $L \simeq \mathbb{R}^2 \times S^1 \hookrightarrow X$; these are often referred to as “toric branes” and were constructed in [7] generalizing [52, 75]. In terms of the web diagram, L is constructed as the co-normal bundle over a straight semi-infinite line in the toric polytope, intersecting one of its edges. To visualize it, let v be a vertex of the toric web and l_i , $i = 1, \dots, 3$ be the three outgoing edges,¹ and let L intersect the edge l_3 . In local co-ordinates x_1, x_2, x_3 dual to the faces l_{23}, l_{13} and l_{12} of the web, the Lagrangian L is the fixed locus of the anti-holomorphic involution $\sigma : X \rightarrow X$ sending

$$\sigma : (x_1, x_2, x_3) \rightarrow (\bar{x}_3 \bar{x}_2, \bar{x}_3 \bar{x}_1, 1/\bar{x}_3). \quad (2.1)$$

The S^1 inside L is the equator $|x_3| = 1$ of the x_3 direction, whereas the \mathbb{R}^2 factor comes from the two real fibers in the transverse direction as in (2.1).

2.1.2. A-model open string instantons. The type A open topological string on (X, L) deals with the (virtual) count of the number of A-model instantons from a genus g , h -holed worldsheet $\Gamma_{g,h}$ to X , with Dirichlet boundary conditions given by L . From a geometric point of view, the natural object to parameterize such instantons would be

¹ We keep supposing that X is smooth, hence all vertices of the toric web are trivalent.

ideally given by a suitable compactification of a moduli space $\mathcal{M}_{g,h}(X, L, \beta, \{d_i\}_{i=1}^h)$ of maps $\phi : \Gamma_{g,h} \rightarrow X$, such that ϕ is holomorphic in the bulk and is continuous along the boundary $\partial\Gamma_{g,h} \subset L$, in a fixed topological sector specified by $\phi_*[\Gamma_{g,h}] = \beta \in H_2(X, L, \mathbb{Z})$ and $\phi_*[D_i] = d_i \in H_1(L) \simeq \mathbb{Z}$. The genus g , h -holes contribution to the open topological string partition function $Z^{X,L}$ would then be given as

$$(\log Z^{X,L})_{g,h}(\mathbf{t}, \mathbf{w}) = F_{g,h}^{X,L}(\mathbf{t}, \mathbf{w}) = \sum_{\beta \in H_2(X, L, \mathbb{Z})} \sum_{d_1, \dots, d_h \in \mathbb{Z}} N_{g,h,\beta,d_1,\dots,d_h}^{X,L} e^{t \cdot \beta} \prod_{i=1}^h w_i^{d_i}, \quad (2.2)$$

where “open Gromov-Witten invariants” $N_{g,h,\beta,d_1,\dots,d_h}^{X,L}$ would be defined as

$$N_{g,h,\beta,d_1,\dots,d_h}^{X,L} \stackrel{!}{=} \int_{[\overline{\mathcal{M}}_{g,h}(X, L, \beta, \{d_i\}_{i=1}^h)]^{\text{vir}}} 1. \quad (2.3)$$

The definition, and *a fortiori* the calculation of $N_{g,h,\beta,d_1,\dots,d_h}^{X,L}$ in (2.3) hinges on finding a suitable Kontsevich-like compactification of $\overline{\mathcal{M}}_{g,h}(X, L, \beta, \{d_i\}_{i=1}^h)$ and on the construction of a top-dimensional homology cycle - the *virtual fundamental class* [10] - with the properties expected from deformation theory. While the conceptual discussion for the open topological A-model would parallel the ordinary closed string case, there are however several important technical points where the open case departs from the one without D -branes.

To start with, the real condition imposed by the Lagrangian forces the open string moduli spaces to be essentially non-(complex)-algebraic, and makes it more difficult to find a viable mathematical compactification of the moduli space as compared to the case of closed strings, let alone the construction of a virtual fundamental class. As emphasized in the foundational work [78], the construction of open string moduli spaces for (X, L) naturally leads to problems coming from the non-orientedness of $\overline{\mathcal{M}}_{g,h}(X, L, \beta, \{d_i\}_{i=1}^h)$, and to the fact that it has a non-trivial boundary in real co-dimension one, thus making open string insertions in principle ill-defined at the level of co-homology.²

One possible way to circumvent this problem and define operatively the invariants is to use localization, an approach put forward in the work of Katz and Liu [59]. Recall that in the closed string case we do have (at least in positive degree) a well-defined construction of a virtual fundamental cycle $[\overline{\mathcal{M}}_g(X, \beta)]^{\text{vir}}$ for the moduli space of stable maps to X , for example in terms of relative stable maps to a projective compactification of X . A torus action $T \times X \rightarrow X$ on X pulls back to an action on $\overline{\mathcal{M}}_g(X, \beta)$; the fundamental cycle $[\overline{\mathcal{M}}_g(X, \beta)]^{\text{vir}}$ induces a T -equivariant virtual cycle $[\overline{\mathcal{M}}_g^T(X, \beta)]^{\text{vir}}$ on the T -fixed locus of $\overline{\mathcal{M}}_g(X, \beta)$. Closed string Gromov-Witten invariants are then computed as

$$N_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_g^T(X, \beta)]^{\text{vir}}} 1 = \sum_i \int_{[\gamma_i, g, \beta]^{\text{vir}}} \frac{1}{e_T(N_{\gamma_i, g, \beta}^{\text{vir}})}, \quad (2.4)$$

² The reader is referred to [78], where these problems are addressed in the case of branes described as the fixed locus of an anti-symplectic involution. More specifically, for the toric case with Aganagic-Vafa branes, see [57] for a different symplectic geometry approach assuming Gromov compactness, and [68] for an algebraic definition in terms of moduli spaces of relative stable morphisms.

where we denoted by $\gamma_{i,g,\beta}$ the fixed components of the T -action. In the presence of a torus action $T \simeq \mathbb{C}^*$ compatible with the anti-holomorphic involution defining L , an extension of this line of reasoning to the open string setting was given in [59]. The authors propose a natural tangent/obstruction theory for the moduli space of open stable maps; the relevant exact sequence reads, in terms of fibers at a smooth point (Γ, f) ,

$$\begin{aligned} 0 \rightarrow H^0(\Gamma, \partial\Gamma, T_\Gamma, T_{\partial\Gamma}) \rightarrow H^0(X, L, f^*T_X, (f|_{\partial\Gamma})^*T_L) \rightarrow \mathcal{T}^1 \rightarrow \\ H^1(\Gamma, \partial\Gamma, T_\Gamma, T_{\partial\Gamma}) \rightarrow H^1(X, L, f^*T_X, (f|_{\partial\Gamma})^*T_L) \rightarrow \mathcal{T}^2 \rightarrow 0. \end{aligned} \quad (2.5)$$

The resulting moduli space, when X is a CY3 and L the fixed locus of an anti-holomorphic involution, has expected dimension zero. If we now *assume* that

1. there is a well-defined $T \simeq \mathbb{C}^*$ action on the moduli space, so that localization theorems apply;
2. we can identify the T -fixed loci $\gamma_{i,g,\beta,\{d_i\}_{i=1}^h}$, and have a natural proposal for the localization of the fundamental cycle $1_{\gamma_{i,g,\beta,\{d_i\}_{i=1}^h}}^{\text{vir}}$,

then the open Gromov-Witten invariants (2.3) can be *defined* by localization

$$N_{g,h,\beta,d_1,\dots,d_h}^{X,L} := \sum_i \int_{[\gamma_{i,g,\beta,\{d_i\}_{i=1}^h}]^{\text{vir}}} \frac{1}{e_T(N_{\gamma_{i,g,\beta,\{d_i\}_{i=1}^h}}^{\text{vir}})}. \quad (2.6)$$

The point which is harder to prove rigorously is the first. If we assume this, though, (2.6) yields an operative definition of the open A-model on (X, L) . In particular, in the case of toric backgrounds with Aganagic-Vafa branes and in the presence of a Calabi-Yau action T compatible with L , it is easy to determine the topological data that define the localization of the virtual cycle to the T -fixed loci. As the choice of the torus T is non-unique, but rather depends on an integer ambiguity $f \in \mathbb{Z}$, the resulting open string invariants depend on an additional \mathbb{Z} -valued parameter

$$N_{g,h,\beta,d_1,\dots,d_h}^{X,L} = N_{g,h,\beta,d_1,\dots,d_h}^{X,L}(f). \quad (2.7)$$

This fact is entirely expected from string duality, as it corresponds to the large N dual incarnation of the framing ambiguity of Wilson loops of knots and links in Chern-Simons theory [75, 79].

In Sect. 3.1.2 we will review the structure of the Atiyah-Bott computations behind (2.6). To conclude this section, let us just mention the case where X is not smooth. The picture can in fact be generalized to include singular toric Calabi-Yau threefolds [13, 14, 17]. In particular, for G finite abelian, let $X = [\mathbb{C}^3/G]$ be a toric Calabi-Yau orbifold of flat space; we choose the fibers x_i , $i = 1, 2, 3$ to carry irreducible representations of the G -action. As the anti-holomorphic involution (2.1) is compatible with a Calabi-Yau G -action, it descends to the quotient defining a Lagrangian $L \subset \mathbb{C}^3/G$; in the presence further of a compatible T -action, localization can be applied to define/compute open orbifold Gromov-Witten invariants of the pair (X, L) in this more general case.

2.2. The equivariant closed string side. As we will see, closed string localization formulae will play a crucial role in what follows; we will then be interested in the problem of computing T -equivariant Gromov-Witten invariants of X . Since turning on a torus action leads to a number of new interesting phenomena with respect to the ordinary non-equivariant case, we briefly review them here.

2.2.1. The T -equivariant A-model. At a worldsheet level, the equivariant integration on the closed string moduli spaces $\mathcal{M}_g(X, \beta)$ is realized as follows. Since X is a toric threefold, we have $\text{rank}_{\mathbb{C}} \text{Iso}(X) = 3$. Whenever the target space possesses a flavor symmetry in the form of a holomorphic isometry, this can be used to generate a $\mathcal{N} = (2, 2)$ potential deformation of the original worldsheet theory [9, 64, 65]. Let ϕ^i be local charts on X , $V \in \mathfrak{iso}(X)$ and write $V = V^i \partial_{\phi_i}$ in components; we will write $T \subset \text{Diff}(X)$ for the abelian flow generated by V . In terms of worldsheet fields in the untwisted theory the deformation reads

$$\delta_T L = -g_{i\bar{j}} |\lambda|^2 V^i \bar{V}^{\bar{j}} - \frac{i}{2} \left(g_{i\bar{i}} \partial_{\bar{j}} V^i - g_{j\bar{j}} \partial_i V^j \right) \left(\lambda \psi_{-}^{\bar{i}} \psi_{+}^j + \text{h.c.} \right), \quad (2.8)$$

where the complex masses λ are the equivariant parameters of the T -action.

This deformation has a series of important consequences. The theory has a modified $\mathcal{N} = (2, 2)$ supersymmetry with a non-vanishing central extension given by Lie_V ; as a consequence, in the A-topologically twisted theory the BRST differential Q is deformed to the equivariant de Rham differential $d - \sqrt{2}i\lambda i_V$, hence squaring to $Q^2 = 2i\lambda \text{Lie}_V$.

A-model chiral operators \mathcal{O}_{α} from the σ -model sector of the theory are now in one to one correspondence to *invariant* forms $\mathcal{O}_{\alpha} \leftrightarrow \phi_{\alpha} \in H_T^{\bullet}(X)$. At the level of the corresponding moduli space of classical trajectories, chiral n -point functions are computed as

$$\langle \mathcal{O}_{\alpha_1} \dots \mathcal{O}_{\alpha_n} \rangle_{g,n,\beta}^{X_T} = \sum_i \int_{[\gamma_{i,g,\beta,n}]^{\text{vir}}} \frac{\prod_{i=1}^n \text{ev}_i^* \phi_{\alpha_i}}{e_T(N_{\gamma_{i,g,\beta,n}}^{\text{vir}})}, \quad (2.9)$$

where the moduli spaces parametrize stable maps from n -pointed curves, with markings corresponding to chiral insertions; we wrote $\text{ev}_i : \mathcal{M}_{g,n}(X, \beta) \rightarrow X$ for the evaluation morphism at the i^{th} marked point

$$\text{ev}_i([\Gamma, f, p_1, \dots, p_n]) = f(p_i). \quad (2.10)$$

As for the ordinary A-model, the resulting chiral ring is an α' -deformation of the T -equivariant de Rham co-homology $H_T^{\bullet}(X)$, called the *big quantum co-homology ring*. When we want to emphasize the stringy deformation of the ring structure, we will write $QH_T^{\bullet}(X)$ to denote the T -equivariant chiral ring.

As a further comment, notice that the extra term (2.8) results in a deformation of the worldsheet theory away from conformality, giving new (twisted) mass terms for the fermions. Moreover, the new fermion mass terms have charge 2 under the A-model ghost number charge, whose R-symmetry is thus broken explicitly by (2.8). From a space-time point of view, the resulting topological string yields refined invariants, counting the number of wrapped $M2$ branes of definite charge under the T -action (see [6] for a discussion, as well as a large N dual description in Chern-Simons theory).

2.2.2. Recoupling to the observables of topological gravity. Gauging the torus action results in an important new extra feature [51, 55]: in the full topological string, we have an infinite tower of non-trivial BRST-closed descendants $\mathcal{O}_{\alpha,p}$ for each \mathcal{O}_{α} , involving the observables of the gravitational sector of the theory (see e.g. [28, 80]). They have the form

$$\mathcal{O}_{\alpha,p} = \sigma^p \phi_{\alpha}, \quad (2.11)$$

where σ is the superfield obtained by topological descent equations on a bosonic operator $\sigma^{(0)}$ given in conformal gauge by

$$\sigma^{(0)} = \frac{1}{2} (\partial\gamma + \gamma\partial\phi - c\partial\psi - \text{h.c.}) \quad (2.12)$$

in terms of the ghost β , γ , the 2D gravitino ψ , and the Liouville field ϕ . Their moduli space realization is given in terms of powers of tautological classes [63]: this leads to a \mathbb{Z}^n -family of n -point chiral *gravitational* correlators

$$\langle \mathcal{O}_{\alpha_1, p_1} \cdots \mathcal{O}_{\alpha_n, p_n} \rangle_{g, n, \beta}^{X_T} = \sum_i \int_{[\gamma_{i, g, n, \beta}]^{\text{vir}}} \frac{\prod_{i=1}^n \text{ev}_i^* \phi_i \psi^{p_i}}{e_T(N_{\gamma_{i, g, n, \beta}}^{\text{vir}})}, \quad (2.13)$$

obtained by capping the pull-backs at the i^{th} marked point of ϕ_α with powers of the i^{th} tautological class $\psi_i = c_1(\mathbb{L}_i)$, where \mathbb{L}_i is the line bundle on $\overline{\mathcal{M}}_{g, n}(X, \beta)$ whose fiber over a smooth moduli point $(f, \Gamma, p_1, \dots, p_n)$ is the cotangent line $T_{p_i}^* \Gamma$.

In the ordinary Calabi-Yau case, this infinite set of gravitational operators is largely decoupled, and at any rate it does not contain any new information with respect to the partition function: $U(1)_R$ charge conservation forces this type of insertions to be mostly zero, or to be trivially proportional to the free energy. In the T -equivariant case, instead, the Calabi-Yau selection rules that “decoupled” topological gravity are violated by terms proportional to the mass terms in (2.8), namely, the equivariant parameter $\lambda = c_1(\mathcal{O}_{\mathbb{CP}^\infty}(1)) \in H_T(\{\text{pt}\})$ of $T \times X \rightarrow X$. This means that the gravitational correlator (2.13) is now generically non-vanishing and carries extra information (proportional to the T -generated mass terms) with respect to the partition function. As a consequence, the equivariant topological A-model on a toric Calabi-Yau threefold closely resembles the topological string in the asymptotically free case, such as the A-model on Fano target manifolds.

As for the ordinary non-equivariant case, it is convenient to pack together equivariant Gromov-Witten invariants inside generating functions. To this aim, we first of all introduce chemical potentials $t_{\alpha, p}$ dual to insertions of $\mathcal{O}_{\alpha, p}$ and we write the genus g , full-descendant equivariant A-model free energy as

$$\mathcal{F}_g^{X, T}(t_{\alpha, p}) = \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{n=0}^{\infty} \sum_{\substack{\alpha_1, \dots, \alpha_n \\ p_1, \dots, p_n}} \frac{\prod_{i=1}^n t_{\alpha_i, p_i}}{n!} \langle \mathcal{O}_{\alpha_1, p_1} \cdots \mathcal{O}_{\alpha_n, p_n} \rangle_{g, n, \beta}^{X_T}, \quad (2.14)$$

where $\alpha = 1, \dots, \chi(X)$ and $p_i \in \mathbb{Z}^+$ for all i ; we use the fact that $\dim_{\mathbb{C}(\lambda)} H_T(X) = \chi(X)$ in the toric case, due to the vanishing of the co-homologies in odd degree. The non-equivariant free energies $F_g^X(t_\alpha)$ can be recovered in two ways:³ either by taking $\lambda \rightarrow 0$, or, when the T -action preserves the Calabi-Yau condition, by restricting the insertions to *small quantum co-homology*, namely, to degree 2 primary operators

$$F_g^X(t_\alpha) = \mathcal{F}_g^{X, T}(t_{\alpha, p}) \Big|_{t_{\alpha, p} = t_\alpha \delta_{p, 0} \delta_{\deg \alpha, 2}}. \quad (2.15)$$

³ To be rigorous, both statements hold strictly speaking only when we discard unstable contributions to the free energy and regularize the degree zero terms at tree-level, as the latter are necessarily singular non-equivariantly due to the non-compactness of X .

An important role in our formalism will be played by a restriction of (2.14) to its $g = 0$, $n = 1$ subsector, in the form of Givental's J -function. This is the $H_T^\bullet(X)$ -valued power series

$$J^{X,T} \left(t^1, \dots, t^{\chi(X)}; z \right) := z + t_\alpha \phi^\alpha + \sum_{n=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} \left\langle \mathbf{t}, \dots, \mathbf{t}, \frac{\phi_\alpha}{z - \psi} \right\rangle_{0, n+1, \beta}^{X_T} \phi^\alpha, \quad (2.16)$$

where

$$\mathbf{t} := \sum_{\beta=1}^{\chi_X} t_\beta \phi^\beta. \quad (2.17)$$

Restricting the sum in (2.17) to degree ≤ 2 classes, we obtain the *small* J-function of X .

3. Our Formalism

Our formalism builds directly on the localization approach to the open A-model on toric backgrounds [17, 59]. In the following, we first of all recall the relation between open string amplitudes and closed descendant invariants in the form (1.1), and give new arguments for its general validity, also in singular phases. We then review and build on the connection between the closed equivariant theory and τ -functions of classical integrable hierarchies.

3.1. Localization and an open/closed string duality. In this section we outline the derivation of the master formula (1.1). We first concentrate on the case when X is a toric orbifold of \mathbb{C}^3 , and then generalize (1.1) to an arbitrary open string toric background (X, L) .

3.1.1. Orbifolds. For future utility, before moving to the analysis of open string instantons for this brane setup let us recall the structure of the closed string sector. To start with, suppose X is a Calabi-Yau abelian orbifold of \mathbb{C}^3 , and let $L \hookrightarrow X$ be the Lagrangian defined by (2.1) (see Fig. 3). Consider the following family of $T \simeq \mathbb{C}^*$ actions on X :

$$\begin{aligned} T \times X &\rightarrow X \\ (\mu, x_1, x_2, x_3) &\rightarrow (\mu^f x_1, \mu^{-f - r_{\text{eff}}} x_2, \mu^{r_{\text{eff}}} x_3). \end{aligned} \quad (3.1)$$

In (3.1), $r_{\text{eff}} = \text{l.c.m.}(p_1^{\text{eff}}, \dots, p_r^{\text{eff}})$, where we decomposed $G = \times_{i=1}^r \mathbb{Z}_i$ and we denoted by $p_i^{\text{eff}} = \text{ord}(\mathbb{Z}_i^{\text{eff}})$ the order of the maximal subgroup of the i^{th} \mathbb{Z}_i factor acting effectively along the x_3 fiber. The framing parameter f can in principle be rational when $G \neq e$, with a denominator that divides r_{eff} .

The T -equivariant chiral ring in this case coincides classically [21, 82] with the T -equivariant orbifold co-homology $H_{\text{orb}, T}^\bullet(X)$,

$$H_{\text{orb}, T}^\bullet(X) = \bigoplus_{g \in G} H_T^\bullet(X_g), \quad (3.2)$$

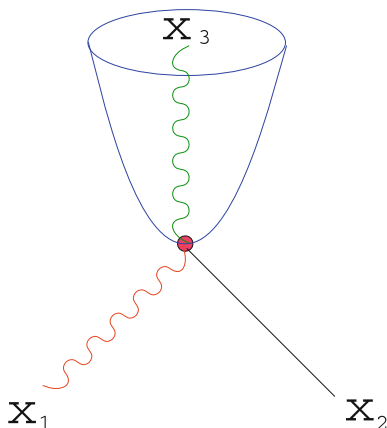


Fig. 3. The orbifold vertex $X = [\mathbb{C}^3/G]$, depicted with two gerby lines along x_1 and x_3 , and a Lagrangian intersecting the equator of the x_3 fiber

where $X_g = \{(x, g) | x \in \mathbb{C}^3\}$ denotes the g -twisted sector of X for $g \in G$. We write $\mathcal{I}X := \bigsqcup_{g \in G} X_g$ for their disjoint union - the *inertia stack* of X - and $\mathbf{1}_g$ for a G -twisted⁴ class in $H_{\text{orb}, T}^\bullet(X)$. The topological two point function is given by the T -equivariant orbifold Poincaré pairing [21],

$$\eta(\mathbf{1}_g, \mathbf{1}_{g'}) = \int_{\mathcal{I}X^T} \frac{\mathbf{1}_g \cup \mathbf{1}_{(g')^{-1}}}{e(N_{\mathcal{I}X^T/\mathcal{I}X})}. \quad (3.3)$$

The A-model ghost number charge in a twisted sector has a contribution coming from the vacuum fermionic shift that affects the orbifold topological string in the presence of g -twisted boundary conditions [82]. Without loss of generality, suppose x_i , $i = 1, 2, 3$ carry one-dimensional irreps of G and denote with $\alpha_{g,i}^r$ the character of $g = (g_1, \dots, g_r) \in \times_{i=1}^r \mathbb{Z}_i$ on the line parameterized by x_i . Then the fermionic shift is equal to twice the *age* of $\mathbf{1}_g$ [21, 82]

$$\text{age}(\mathbf{1}_g) = \sum_{i=1}^3 \sum_{j=1}^r \frac{\alpha_{g,i}^j}{p_j^{\text{eff}}} \in \mathbb{Z} \quad (3.4)$$

and we have for the orbifold (or Chen-Ruan) degree

$$\deg(\mathbf{1}_g) = 2\text{age}(\mathbf{1}_g). \quad (3.5)$$

3.1.2. The master formula for open string invariants: the orbifold vertex. Open string instantons of (X, L) have the following structure. The T -fixed points inside $\overline{\mathcal{M}}_{g,h,n}(X, L, \beta, \{d_i\}_{i=1}^h)$ consist of a compact genus g curve, carrying n twisted marked points, with a collection of h (orbi-)discs attached, as depicted in Fig. 4. The compact curve contracts to the vertex of the toric diagram, and the discs are mapped (rigidly) to the lower hemisphere $|x_3| < 1$ of the compactified x_3 fiber, with their boundary wrapping around the equator.

⁴ We hasten to warn the reader that by “twisted” we do *not* mean “twisted chiral”: all the discussion is strictly holomorphic here. In this section, “twisted” refers to the fact that these insertions come from the twisted sectors of the orbifold topological string.

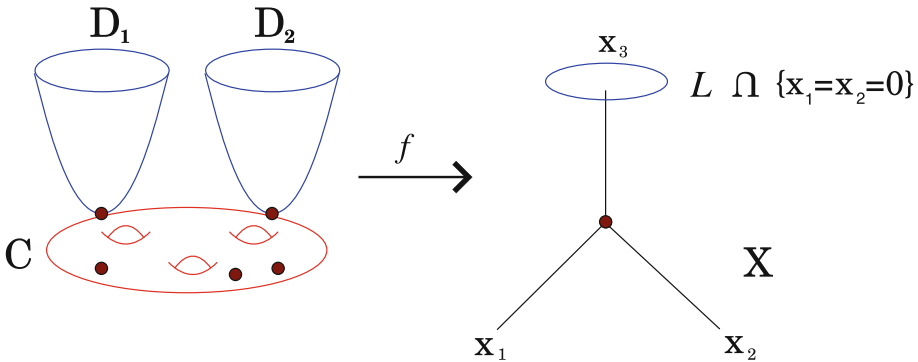


Fig. 4. A pictorial view of a torus-fixed stable map from a genus $g = 3, h = 2$ (orbi)-curve to $X = [\mathbb{C}^3/G]$ (from [17])

The restriction of the Katz-Liu obstruction theory (2.5) to a T -fixed locus consists of essentially three pieces; the reader is referred to [17, 59] for more details. Suppose we only have primary (matter) insertions $\phi_{\alpha_i} \in H_{\text{orb}}^\bullet(X)$, $i = 1, \dots, n$, of Chen-Ruan degree 2 at the n marked points. The contracting compact curve and the n marks yield a factor

$$\prod_{i=1}^3 \Lambda_i(\mu_i) \prod_{j=1}^n \text{ev}_j^* \phi_{\alpha_j} \in H^{\text{top}}(\overline{\mathcal{M}}_{g,n}(BG, 0), \mathbb{Q}) \quad (3.6)$$

corresponding to the dual of an n -pointed insertion that contributes to the closed equivariant genus g free energy; in (3.6), $\Lambda_i(\mu_i)$ is the T -equivariant Euler class of the dual of an appropriate sub-bundle of the Hodge bundle [17], linearized with the weights of the torus action $\mu_1 = -f - r_{\text{eff}}$, $\mu_2 = f$, $\mu_3 = r_{\text{eff}}$ as in (3.1). The i^{th} node contribution brings about a gravitational contribution of the type $\frac{1}{\frac{\lambda}{d_i} - \psi}$ for each node i , as well as a universal constant normalization term [17]; finally, each disc contributes a factor $D_{\alpha}^{X,L}(d_i, f)$, which is completely determined by the brane setup and the choice of framing. Altogether, and taking into account [17] the compatibility condition between degree of the map and twisting at the nodes, we obtain that a genus g, h -holed amplitude on (X, L) is given by⁵

$$F_{g,h}^{X,L}(t_1, \dots, t_n; w_1, \dots, w_h; f) = \left(\prod_{i=1}^h \mathcal{L}^{X,L}(w_i, f) \right) \mathcal{F}_g^X(t_{\alpha,p}, f) \Big|_{t_{\alpha,p} = \delta_{\deg \phi_{\alpha}, 2\delta_{p,0}t_{\alpha}}}, \quad (3.7)$$

where the *brane insertion operator* $\mathcal{L}^{X,L}(w, f)$ has the form

$$\mathcal{L}^{X,L}(w, f) = \sum_{n=0}^{\infty} \sum_{d=1}^{\infty} \sum_{\alpha \in H_{\text{orb}}^\bullet(X)} w^d d^{n+1} D_{\alpha}^{X,L}(d, f) \frac{\partial}{\partial t_{\alpha,n}} \quad (3.8)$$

⁵ In some conventions [50], $F_{g,h}^{X,L}$ is an element in $H_T^{2h}(\text{pt})$; what we call $F_{g,h}^{X,L}$ here is the coefficient of proportionality of λ^h there.

and $\mathcal{F}_g^X(\tau_{\alpha,p}, f)$ is the full-descendant genus g free energy (2.14). This is what we called the “master formula” for open string amplitudes in (1.1).

The disc contribution $D_\alpha^{X,L}(d, f)$ is what specifies the form of the brane insertion operator $\mathcal{L}^{X,L}(w, f)$ for a given open string geometry, and can be computed directly by localization. For instance, when $G = \mathbb{Z}_p$ and denoting chiral insertions from twisted sectors by $\mathbf{1}_k, k \in \mathbb{Z}_p$, we have

$$D^{X,L}(d, f) =: \sum_{k=0}^{p-1} D_k^{X,L}(d, f) \mathbf{1}_k, \quad (3.9)$$

$$D_k^{X,L}(d, f) = \left(\frac{1}{d}\right)^{\text{age}(\mathbf{1}_k)} \frac{1}{\lfloor \frac{d}{r_{\text{eff}}} \rfloor!} \frac{\Gamma\left(d\mu_1 + \left\langle \frac{k\alpha_2}{n} \right\rangle + \frac{d}{r_{\text{eff}}}\right)}{\Gamma\left(d\mu_1 - \left\langle \frac{k\alpha_1}{p} \right\rangle + 1\right)},$$

where we wrote α_i for the characters of the \mathbb{Z}_p action along the i^{th} leg of the vertex.

Let us examine (3.7) more closely. According to this formula, the open topological A-model on a background with toric branes is controlled by a dual closed theory on the same background with gravitational descendants turned on. What is more, the precise relationship is given through a concrete incarnation of a crucial object in the theory of matrix models, namely the loop insertion operator. The relation between the open and the closed model has indeed the same structure as the one between connected correlators and deformed free energies in matrix models [8], and is in complete agreement with the mirror description in terms of B-branes [2]: the open topological free energy is obtained by the action of a 1st-order differential operator in an infinite number of new modes on a deformed closed amplitude. In our language, the Ooguri-Vafa operator would take the form

$$e^{\mathcal{L}} := \sum_{h=0}^{\infty} g_s^h \prod_{i=1}^h \mathcal{L}(w_i, f), \quad (3.10)$$

$$F^{\text{open}} = e^{\mathcal{L}} \mathcal{F}^{\text{closed, desc}} \Big|_{\text{s.q.c.}}, \quad (3.11)$$

where we denoted by $\mathcal{A}|_{\text{s.q.c.}}$ the reduction to small quantum co-homology. On the other hand, the insertion of a toric brane - or more precisely, in B-model language, the insertion of a determinant in the mirror Kodaira-Spencer theory [2] - can be recast in the form of a gravitational background shift as

$$t_{\alpha,p} \rightarrow t_{\alpha,p} + \sum_{d=1}^{\infty} D_\alpha^{X,L}(d, f) w^d d^{p+1}, \quad (3.12)$$

where w is the A-model open string modulus.

It is instructive to look at the particular case $g = 0, h = 1$. When $g = 0, h = 1$, Eq. (3.7) states that the winding number d contribution to the disc amplitude takes the compact form

$$\oint_{w=0} \frac{1}{2\pi i w^{d+1}} F_{0,1}^{X,L}(t_1, \dots, t_n; w, f) = D_\alpha^{X,L}(d, f) J^\alpha \left(t_1, \dots, t_n; f; \frac{1}{d} \right) \quad (3.13)$$

in terms of the T -equivariant J -function (2.16) of X .

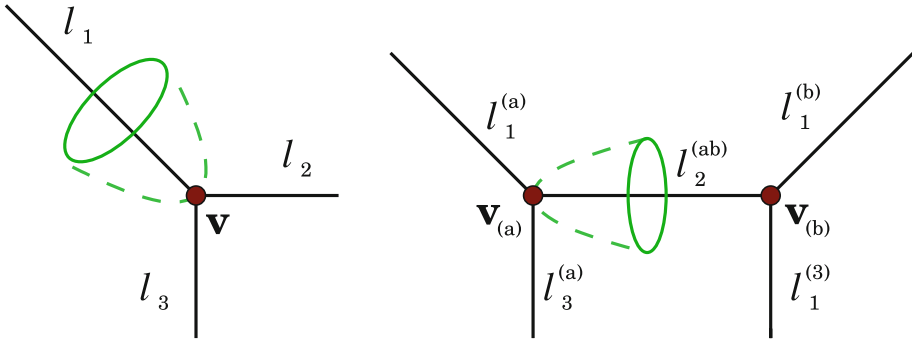


Fig. 5. A toric brane ending on an external leg (left) and an internal leg (right) of the web diagram

3.1.3. General toric Calabi-Yau threefolds. Up to now we have only considered a particular case of toric Calabi-Yau threefolds, namely, toric orbifolds of \mathbb{C}^3 . However, we claim that the master formula (3.7) holds true when X is a general TCY3, and in any patch of its stringy moduli space.

A first way to see this, and a more natural one from the point of view of localization, is that the computation of open string invariants for toric branes ending on a vertex of the web diagram of X is essentially a local operation.⁶ As explained in [50], the bulk geometry affects the open string amplitude only by replacing (3.6) by a term corresponding to primary closed string insertions - i.e., for local geometries given by neighbourhoods of a rigid curve or surface $\Sigma \hookrightarrow X$, by the push-pull of the normal bundle to Σ on $\overline{\mathcal{M}}_{g,n}(\Sigma, \beta)$, capped with pull-backs of co-homology classes ϕ_{α_i} , $i = 1, \dots, n$ at the n insertion points. On the other hand, the local geometry is entirely controlled by the vertex computation of the previous section, with the (possibly G -twisted) chiral operators of the local theory that are lifted to (possibly G -twisted) operators in the full chiral ring of X . Two cases are possible; we restrict here the discussion to the case in which X is smooth. Suppose first that the leg l_1 on which the brane ends is external (see Fig. 5), and let v be the tri-valent vertex to which it is connected; we will call $C_v \simeq \mathbb{C}^3$ the affine patch of X associated to the vertex v . Then the loop insertion operator (3.8) for this setup has the same form with a disc contribution $D_{\alpha}^{X,L}(d, f)$ given by

$$D_{\alpha}^{X,L}(d, f) = D_{\text{id}}^{C_v \simeq \mathbb{C}^3, L}(d, f) \delta_{\alpha, \alpha_v}, \quad (3.14)$$

where $\phi_{\alpha_v} \in H_T(X)$ is the equivariant class of the “tip” of the disc attaching at the origin of C_v [50]. When the brane intersects an inner leg, we have two fixed vertices $v^{(a)}$ and $v^{(b)}$ the marked point can attach to (Fig. 5). Then

$$D_{\alpha}^{X,L}(d, f) = D_{\text{id}}^{C^3, L}(d, f) (\delta_{\alpha, \alpha_{v^{(a)}}} + \delta_{\alpha, \alpha_{v^{(b)}}}). \quad (3.15)$$

3.1.4. Moving in the open and closed moduli space. There is a second vantage point to look at (3.7) for a general pair (X, L) , which is motivated by open string mirror symmetry.⁷ The derivative of the B-model disc amplitude, which captures the (infinitesimal) domain-wall tension of a $D5$ brane wrapping the mirrors of toric branes, should be a

⁶ We thank Renzo Cavalieri for enlightening discussions of this point.

⁷ For a different point of view, see also the very recent preprint [20].

holomorphic, globally defined function of the vector multiplets [13, 53]. Let \hat{X} denote the mirror of X , $\mathcal{M}_{\hat{X}}$ be the natural toric compactification of its complex moduli space [26], and $\mathfrak{c} \subset \mathcal{M}_{\hat{X}}$ be a chamber in $\mathcal{M}_{\hat{X}}$. Also in view of [17], it is natural to speculate that the B-brane superpotential in a specific chamber should take the form

$$\sum_{d=1}^{\infty} D^{(\mathfrak{c})}(d, f) \cdot I^{(\mathfrak{c})}\left(\mathbf{x}; \frac{1}{d}, f\right) y^d, \quad (3.16)$$

where y is the B-model open modulus y , $I^{(\mathfrak{c})}(\mathbf{x}; \cdot, f)$ is Givental's I -function in chamber \mathfrak{c} [25, 47], and \mathbf{x} are B-model co-ordinates around the relevant boundary point. The I -function is a co-homology valued generalized hypergeometric series in the variables \mathbf{x} , whose form we can read off from the fan \mathcal{F}_X (see Appendix A), and whose components provide a basis of solutions for the T -equivariant Picard-Fuchs system [47] associated to X . The I -function is closely related to the J -function (2.16) restricted to small quantum co-homology, as we will see in a moment; we refer the reader to Appendix A for a detailed account on J and I functions of toric threefolds.

As we emphasized, the disc amplitude is a holomorphic globally defined quantity; on the other hand, I -functions in different chambers should be related to one another by analytic continuation and a z -dependent linear automorphism, as they provide bases of solutions of the same holonomic system of PDEs in the \mathbf{x} -variables, namely, the equivariant Picard-Fuchs system [22, 24, 43, 56]. Then

$$I^{(\mathfrak{c}')}(\mathbf{x}; z, f) = M^{\mathfrak{c}\mathfrak{c}'}(z) I^{(\mathfrak{c})}(\mathbf{x}; z, f) \quad (3.17)$$

for some invertible matrix $M^{\mathfrak{c}\mathfrak{c}'}(z)$, and imposing invariance of the disc amplitude we get

$$D^{(\mathfrak{c}')}(\mathbf{x}; d, f) = \left[\left[M^{\mathfrak{c}\mathfrak{c}'}\left(\frac{1}{d}\right) \right]^{-1} \right]^T D^{(\mathfrak{c})}(\mathbf{x}; d, f) \quad (3.18)$$

which expresses the change of the brane insertion operator when moving from one phase to another in the stringy moduli space of X .

Formula (3.18) and the localization formulae for orbifolds \mathbb{C}^3 can be used as an alternative to (3.14) to compute the brane insertion operator for general toric Calabi-Yau threefolds. To see this, notice that any TCY3 is a partial crepant resolution of \mathbb{C}^3/G for some G , perhaps upon taking the limit of infinite Kähler volume for the curves representing some the generators of $H_2(X, \beta)$ (Fig. 6); in particular, in complex dimension three we can choose $G \simeq \mathbb{Z}_p \times \mathbb{Z}_q$ for some $p, q \in \mathbb{Z}$. This can be seen diagrammatically by adding a finite set Θ_X of 1-dimensional cones to the fan \mathcal{F}_X such that the convex hull of the toric diagram is a triangle; the enlarged stringy moduli space now incorporates an orbifold point with enhanced $\mathbb{Z}_p \times \mathbb{Z}_q$ monodromy. The disc function in a specific chamber can be computed starting from the one at such orbifold point, using (3.18) to move to any given chamber of the moduli space of the toric variety associated to the enlarged fan, and finally decoupling the Kähler moduli corresponding to the rays of $\mathcal{F}_X \setminus \Theta_X$ (see Fig. 6). We will verify explicitly in the example of Sect. 4.3 the consistency of (3.18) with (3.14).

The fact that the B-model superpotential is given by folding the disc prefactor $D^{(\mathfrak{c})}(d, f)$ with the I -function in the relevant chamber has another interesting consequence. In the local models under scrutiny, and when restricted to small quantum

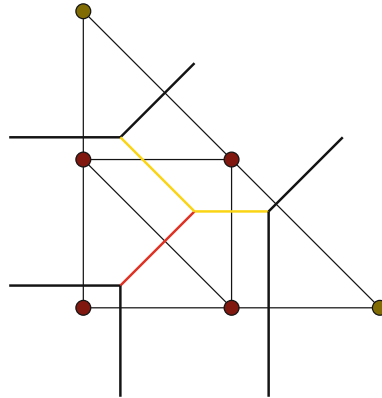


Fig. 6. The minimal $[\mathbb{C}^3/G]$ orbifold for which a toric crepant resolution contains the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ as a decoupling limit. In this case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$: one of the large radius points corresponds to the closed topological vertex geometry, that is, the local CY3 geometry in the neighbourhood of three $(-1, -1)$ \mathbb{P}^1 's touching at a point. In the limit of infinite Kähler volume for two (gold rays/lines) of the three \mathbb{P}^1 , the local geometry reduces to the resolved conifold

co-homology [22,24], the generating function of 1-point descendant Gromov-Witten invariants (2.16) and the I -function in A -model co-ordinates differ by a proportionality factor⁸

$$J^{(c)}(\mathbf{t}; z, f) = f_c(\mathbf{t})^{\frac{1}{z}} I^{(c)}(\mathbf{x}(\mathbf{t}); z, f), \tag{3.19}$$

where $f_c(\mathbf{t})$ is a scalar depending on the closed string moduli only. Then (3.13) and the invariance condition (3.16) readily imply that the A -model open string flat co-ordinates in chambers c and c' are related by a renormalization of the type

$$w_c \rightarrow \frac{f_{c'}(\mathbf{t})}{f_c(\mathbf{t})} w_c = w_{c'}. \tag{3.20}$$

This is in complete agreement with the expectations from mirror symmetry [5, 13, 66], which predict a purely closed string renormalization of open string moduli. As a consequence, the master formula (3.7) automatically embeds the change in open string moduli induced by a closed string phase transition! In Sect. 4.3 we will provide a detailed derivation of this phenomenon for the case of local \mathbb{P}^2 .

3.2. Equivariant Gromov-Witten invariants and integrable hierarchies. In the previous section we showed that (3.7) should hold for a general toric Calabi-Yau threefold with toric A -branes, in all patches of the closed string moduli space, and we discussed in detail what form the brane insertion operator $\mathcal{L}(w, f)$ takes in each case. A point we have not addressed yet is how to compute the second term in (3.7), namely, the T -equivariant free energies of X . We claim that

$$\mathcal{F}_g(t_{\alpha,p}, f) = \left[\ln \tau(t_{\alpha,p} + x\delta_{\alpha,1}\delta_{p,0}, f) \right]_{[g]}, \tag{3.21}$$

⁸ This is a consequence of the fact that both J and I are elements of Givental's Lagrangian cone \mathcal{L}_X , that the cone is invariant under multiplication by a scalar factor, and by uniqueness properties of the J function as an element in \mathcal{L}_X .

where $\tau(t_{\alpha,p}, f, g_s)$ is the τ -function of a $1 + 1$ dimensional Hamiltonian integrable hierarchy of evolutionary PDEs

$$\begin{aligned} u_{t_{\eta,k}}^\alpha &= f_\beta^\alpha(u, \eta, k) u_x^\beta \\ &\quad + g_s^2 \left[g_\beta^\alpha(u, \eta, k) u_{xxx}^\beta + h_{\beta\gamma}^\alpha(u, \eta, k) u_{xx}^{\beta\gamma} + j_{\beta\gamma\delta}^\alpha(u, \eta, k) u_x^{\beta\gamma\delta} \right] + \mathcal{O}(g_s^4) \\ &= \{u^\alpha, H_{\eta,k}[u, g_s]\} = g_s^2 \frac{\partial^2 \log \tau}{\partial x \partial t_{\alpha,k}}, \quad k \in \mathbb{N}, \end{aligned} \quad (3.22)$$

and the subscript $f_{[n]}$ in (3.21) indicates the $(2n - 2)^{\text{nd}}$ coefficient in a Taylor-Laurent expansion in g_s around $g_s = 0$. In other words, the string loop expansion corresponds to a gradient expansion in the x -variable, which is in turn identified with the “puncture” time variable $t_{1,0}$.

In the rest of the section we will first give a concrete description of the hierarchy (3.22) at string tree-level, and then discuss how to deform it to incorporate higher genus corrections. This is in large part a review of known material, with few extra ingredients to take into account the gauged T -action. The reader is referred to [15, 30–32, 77, 81] for more details.

3.2.1. The tree-level hierarchy and genus zero amplitudes. Let us start from the leading order in g_s of (3.22),

$$u_{t_{\eta,k}}^\alpha = f_\beta^\alpha(u, \eta, k) u_x^\beta = \{u^\alpha, H_{\eta,k}[u]_{[0]}\} = g_s^2 \frac{\partial^2 \log \tau_{[0]}}{\partial x \partial t_{\alpha,k}}. \quad (3.23)$$

We want to associate a dynamical system of this form to the planar, T -equivariant A-model on X . The functional space the fields $u^\alpha(x)$ belong to, the (dispersionless) Poisson structure $\{, \}$, and the tower of Hamiltonians $H_{\eta,k}[u]_{[0]}$ are constructed as follows [30, 31, 81]:

1. the phase space $\mathcal{F} \ni u^\alpha(x)$ is given by the loop space $L(H_T(X)) = C^\infty(S^1, QH_T(X))$ of the T -equivariant chiral ring;
2. a Poisson structure $\{, \}$ on \mathcal{F} is induced via an m -component generalization of the KdV poisson bracket as

$$\{u^\alpha(x), u^\beta(y)\} = \eta^{\alpha\beta} \delta'(x - y), \quad \alpha, \beta = 1, \dots, m = \dim_{\mathbb{C}(\lambda)} QH_T(X), \quad (3.24)$$

where $\eta^{\alpha\beta}$ are the coefficients of the inverse of the tt -metric on $QH_T(X)$, that is, the (possibly orbifold) Poincaré pairing on X ;

3. the tree-level Hamiltonians $H_{\alpha,p}[u]_{[0]}$ are local densities

$$H_{\alpha,p}[u]_{[0]} = \int_{S^1} h_{\alpha,p}(u(x))_{[0]} dx, \quad (3.25)$$

where $h^\alpha(u, z)_{[0]} = \eta^{\alpha\beta} h_\alpha(u, z)_{[0]}$ are a system of flat co-ordinates for Dubrovin’s deformed Gauss-Manin connection on $H_T(X)$,

$$\nabla_z = d + \frac{1}{z} \Gamma, \quad \Gamma_{\alpha\beta}^\gamma = \eta^{\gamma\delta} \frac{\partial^3 F_0}{\partial t^\alpha \partial t^\beta \partial t^\delta}, \quad (3.26)$$

and $h_\alpha(u, z)_{[0]} =: \sum_{p \geq 0} h_{\alpha,p}(u)_{[0]} z^{-p}$.

The resulting dynamical system is completely specified by the chiral two-point function η and the A-model Yukawa couplings Γ , both of which are determined by the toric data defining $X^{\circ T}$: the tt -metric can be computed through a degree zero equivariant co-homology computation on X , whereas the Yukawas can be extracted from the large z -expansion of the J -function of X , which is in turn entirely determined by the GLSM charge vectors (see Appendix A).

It can then be proven [30,31] that

1. the resulting Hamiltonians are in involution,

$$\{H_{\alpha,p}[u]_{[0]}, H_{\beta,q}[u]_{[0]}\} = 0 \quad \text{for all } \alpha, \beta, p, q, \quad (3.27)$$

for generic framing, when T acts with compact fixed loci, the family of Hamiltonian conservation laws determined by $H_{\alpha,p}[u]_{[0]}$ is complete;

2. the Hamiltonian flows (3.23) satisfy the τ -symmetry condition

$$\frac{\partial h_{\alpha,p}(u)_{[0]}}{\partial t_{\beta,q}} = \frac{\partial h_{\beta,q}(u)_{[0]}}{\partial t_{\alpha,p}} = g_s^2 \partial_x \frac{\partial^2 \log \tau_{[0]}}{\partial t_{\alpha,p} \partial t_{\beta,q}} \quad (3.28)$$

in terms of the planar τ -function $\tau_{[0]}$;

3. the logarithm $\mathcal{F}_0 = g_s^2 \ln \tau_{[0]}$ of the genus zero τ -function satisfies the system of PDEs [29,80]

$$\frac{\partial^3 \mathcal{F}_0}{\partial t_{\alpha,p} \partial t_{\beta,q} \partial t_{\gamma,r}} = \frac{\partial^2 \mathcal{F}_0}{\partial t_{\alpha,p} \partial t_{\mu,0}} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_{\nu,0} \partial t_{\beta,q} \partial t_{\gamma,r}} \quad (3.29)$$

as well as the string equation

$$\partial_x \mathcal{F}_0 = \sum_{\alpha,p} t_{\alpha,p} \partial_{t_{\alpha,p-1}} \mathcal{F}_0 + \frac{1}{2} \eta_{\alpha\beta} t_{\alpha,0} t_{\beta,0}. \quad (3.30)$$

We refer the reader to [31] for further details. Constructing the dispersionless hierarchy (3.23) that governs the T -equivariant A-model at tree level just amounts to find a set of flat co-ordinates for the deformed connection $\nabla_z = 0$. The genus zero full-descendent free energy $\mathcal{F}_0(t_{\alpha,p}, f)$ is the potential of the integrability condition (3.28), associated to an orbit specified by

$$u_{\alpha}(x) \Big|_{\substack{t_{\beta,0}=t^{\beta} \\ t_{\beta,p}=0 \quad p>0}} = t_{\alpha} + x \delta_{\alpha,1}. \quad (3.31)$$

We will see in Sect. 4 a few concrete examples of this procedure.

We conclude this section with two remarks. First of all, it is natural to ask how much of this setting goes through to the ordinary, un-gauged case; in fact, consistently with the discussion in Sect. 2.2, this case corresponds to a singular limit in which (3.23) becomes ill-defined, as the inverse tt -metric in (3.24) vanishes due to non-compactness of the target space. Secondly, notice that knowledge of the T -equivariant J -function of X allows to fully reconstruct the descendent theory, by employing the genus zero topological recursion relations (3.29) and the string equation (3.30). In particular, the generating function of two-point gravitational descendants with primary insertions

$$\left\langle \left\langle \frac{\phi^{\alpha}}{z - \psi}, \frac{\phi^{\beta}}{y - \psi} \right\rangle \right\rangle_0^{X_T}(\mathbf{t}) := \frac{1}{zw} \sum_{l,m,n=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} \langle \mathcal{O}_{\alpha,l}, \mathcal{O}_{\beta,m}, \mathbf{t}^n \rangle_{0,2+n,\beta}^{X_T} z^{-l} w^{-m}, \quad (3.32)$$

which by (3.8) and (3.7) is the closed string amplitude controlling the annulus function on the open side, is computed in terms of the J -function as [31]

$$\left\langle\left\langle \frac{\phi^\alpha}{z-\psi}, \frac{\phi^\beta}{y-\psi} \right\rangle\right\rangle_0^{X_T}(\mathbf{t}) = \frac{1}{z+w} \left(\partial_\mu \left\langle\left\langle \frac{\phi^\alpha}{z-\psi} \right\rangle\right\rangle_0^{X_T}(\mathbf{t}) \partial^\mu \left\langle\left\langle \frac{\phi^\beta}{w-\psi} \right\rangle\right\rangle_0^{X_T}(\mathbf{t}) - \eta^{\alpha\beta} \right). \quad (3.33)$$

A word of caution is in order to compute the r.h.s. of (3.33). To determine the derivative $\partial_\mu J$ of the J function we should know the expression of the latter in *big* quantum co-homology - namely, the quantum parameter \mathbf{t} should take arbitrary values in $QH_T(X)$. On the other hand, the Coates-Givental formulae (A.7), (A.9) we will use in our applications will only provide us with the restriction of the J -function of X_T to *small* quantum co-homology. However, since the small quantum co-homology ring generates multiplicatively the entire chiral ring in the toric case, knowledge of the small J -function is sufficient to compute all the derivatives $\partial_\mu J$ in (3.33): two-pointed invariants with one primary insertion can be computed from the derivatives of the small J -function using

$$z \nabla_\alpha \nabla_\beta J = \nabla_{\alpha \star \beta} J, \quad (3.34)$$

where $\alpha \star \beta$ in (3.34) denotes the operator product of chiral observables ϕ_α, ϕ_β . Higher order amplitudes can be computed similarly from (3.29).

3.2.2. Quasi-Miura transformations and higher genus corrections. Having associated a quasi-linear hierarchy to the planar A-model, the next and more difficult task is to incorporate higher derivative (i.e., string loop) corrections to (3.23). A strategy to perform this task in the context of asymptotically free topological σ -models coupled to gravity was proposed in [32], to which we refer the reader for an extensive discussion. In [32], the problem of adding higher genus corrections is cast in terms of a g_s -dependent redefinition of the fields

$$u^\alpha(x)_{[0]} \rightarrow u^\alpha(x, g_s) = g_s^2 \frac{\partial^2 \mathcal{F}}{\partial x \partial t_{\alpha,0}}(x, g_s) \quad (3.35)$$

which sends the dispersionless hierarchy (3.23) to its full-dispersive completion (3.22). The sought-for change of dependent variables (3.35) should be a *rational* (or *quasi*-) *Miura transformation*: namely, in our τ -symmetric context, it takes the form

$$\frac{\mathcal{F}_0(u)}{g_s^2} \rightarrow \mathcal{F}[u] = \frac{\mathcal{F}_0(u)}{g_s^2} + \mathcal{F}_1(u, u_x, \dots) + g_s^2 \mathcal{F}_2(u, u_x, \dots) + \mathcal{O}(g_s^4), \quad (3.36)$$

where the coefficient of g_s^n , $n \geq 0$ in the right hand side is a degree n rational function of the field derivatives $u^{(j)}$ for $j > 0$.

The (difficult) task of computing higher genus corrections for X and T can then be viewed as the problem of determining the explicit form of the quasi-Miura transformation (3.36). In the ordinary A-model case,⁹ \mathcal{F}_g can be computed recursively through a set of differential constraints given by the Dubrovin-Zhang loop equation [32]. The equivariant case is however more complicated, and we should argue differently, as in [15].

⁹ More precisely: for the non-equivariant Gromov-Witten theory of manifolds with semi-simple quantum co-homology and vanishing in odd degrees, assuming a suitable form of the Virasoro conjecture [32, 34].

From a practical point of view, we can use the following computational scheme that allows to compute (3.36) at low genus and readily compare with [13]. The basic idea, as in [33], is to exploit the existence of universal relations between co-homology classes on the moduli space of stable maps, which highly constrain the form of the right hand side of (3.36). First of all, the $3g - 2$ theorem of [35, 46] constrains the dependence on field derivatives of \mathcal{F}_g to be of the form

$$\mathcal{F}_g[u] = \mathcal{F}_g \left(u, u_x, u_{xx}, \dots, u^{(3g-2)} \right). \quad (3.37)$$

Secondarily, higher genus analogues of the topological recursion relations (3.29) exist [11, 29, 44, 45, 62, 73]. At low genus ($g \leq 2$), and combined with the $3g - 2$ theorem, they will allow us to recover the (full-descendant) theory in terms of lower genus gravitational n -point functions and degree zero invariants. For example, at genus 1 we have

$$\frac{\partial \mathcal{F}_1}{\partial t_{\alpha,p}} = \frac{\partial^2 \mathcal{F}_0}{\partial t_{\alpha,p-1} \partial t_{\nu,0}} \eta^{\mu\nu} \frac{\partial \mathcal{F}_1}{\partial t_{\mu,0}} + \frac{1}{24} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_{\alpha,p-1} \partial t^\nu \partial t^\mu} \quad (3.38)$$

which determines gravitational n -point couplings as a function of the matter couplings, as well as the tensorial identity on $H_T(X)$,

$$\begin{aligned} 0 = & 3c_{\alpha_1\alpha_2}^\mu c_{\alpha_3\alpha_4}^\nu \frac{\partial^2 \mathcal{F}_1}{\partial t_\mu \partial t^\nu} - 4c_{\alpha_1\alpha_2}^\mu c_{\alpha_3\mu}^\nu \frac{\partial^2 \mathcal{F}_1}{\partial t_{\alpha_4} \partial t^\nu} + 2c_{\alpha_1\alpha_2\alpha_3}^\mu c_{\alpha_4\mu}^\nu - c_{\alpha_1\alpha_2}^\mu c_{\alpha_3\alpha_4\mu}^\nu \frac{\partial \mathcal{F}_1}{\partial t^\nu} \\ & + \frac{1}{6} c_{\alpha_1\alpha_2\alpha_3}^\mu c_{\alpha_4\mu\nu}^\nu + \frac{1}{24} c_{\alpha_1\alpha_2\alpha_3\alpha_4}^\mu c_{\mu\nu}^\nu - \frac{1}{4} c_{\alpha_1\alpha_2\nu}^\mu c_{\alpha_4\mu\alpha_3}^\nu, \end{aligned} \quad (3.39)$$

where

$$\begin{aligned} c_{\alpha\beta}^\mu &= \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t^\alpha \partial t^\beta \partial t^\nu}, & c_{\alpha\beta\gamma}^\mu &= \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\nu}, \\ c_{\alpha\beta\gamma\delta}^\mu &= \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\delta \partial t^\nu}, & F_g(t^\alpha) &= \mathcal{F}_g(t_{\alpha,p}) \Big|_{t_{\alpha,p}=0 \text{ for } p>0, t_{\alpha,0}=t^\alpha}, \end{aligned} \quad (3.40)$$

which determines matter couplings in terms of explicitly calculable degree zero contributions. Analogously, at genus 2 we have two sets of topological recursion relations. Two identities which will suffice for our purposes are

$$\begin{aligned} \frac{\partial \mathcal{F}_2}{\partial t_{\alpha,p}} = & \frac{\partial^2 \mathcal{F}_0}{\partial t_{\alpha,p-1} \partial t_{\beta,0}} \frac{\partial \mathcal{F}_2}{\partial t_{\gamma,0}} \eta^{\beta\gamma} + \frac{\partial^2 \mathcal{F}_0}{\partial t_{\alpha,p-2} \partial t_{\beta,0}} \eta^{\beta\gamma} \left(\frac{\partial \mathcal{F}_2}{\partial t_{\gamma,1}} - \eta^{\delta\varepsilon} \frac{\partial^2 \mathcal{F}_0}{\partial t_{\gamma,0} \partial t_{\delta,0}} \frac{\partial \mathcal{F}_2}{\partial t_{\varepsilon,0}} \right) \\ & + \eta^{\varepsilon\gamma} \eta^{\beta\delta} \left[\frac{\partial^3 \mathcal{F}_0}{\partial t_{\alpha,p-2} \partial t_{\beta,0} \partial t_{\gamma,0}} \left(\frac{7}{10} \frac{\partial \mathcal{F}_1}{\partial t_{\delta,0}} \frac{\partial \mathcal{F}_1}{\partial t_{\varepsilon,0}} + \frac{1}{10} \frac{\partial^2 \mathcal{F}_1}{\partial t_{\delta,0} \partial t_{\varepsilon,0}} \right) \right. \\ & + \frac{13}{240} \frac{\partial^4 \mathcal{F}_0}{\partial t_{\alpha,p-2} \partial t_{\beta,0} \partial t_{\gamma,0} \partial t_{\delta,0}} \frac{\partial \mathcal{F}_1}{\partial t_{\varepsilon,0}} - \frac{1}{240} \frac{\partial^2 \mathcal{F}_1}{\partial t_{\alpha,p-2} \partial t_{\beta,0}} \frac{\partial^3 \mathcal{F}_0}{\partial t_{\delta,0} \partial t_{\gamma,0} \partial t_{\varepsilon,0}} \\ & \left. + \frac{1}{960} \frac{\partial^5 \mathcal{F}_0}{\partial t_{\alpha,p-2} \partial t_{\beta,0} \partial t_{\gamma,0} \partial t_{\delta,0} \partial t_{\varepsilon,0}} \right] \end{aligned} \quad (3.41)$$

and the Belorusski-Pandharipande equation [11] for the matter free energy, which generalizes (3.39) to two-loops.

From a conceptual view, the all-genus recursive approach of [32] was shown by the authors to be equivalent to Givental's quantization formalism [48]; as the latter applies

also to non-conformal Frobenius structures, as long as semi-simplicity of the quantum product is preserved, it goes through to the case of our interest. This provides in principle a complete solution to the reconstruction of the higher genus theory; yet, its concrete implementation is far from trivial. We hope to report on this problem in detail in the near future. Moreover, in some cases of interest as for example the resolved conifold [15, 16] or configurations of rational curves in a CY3, an all-genus answer can be obtained through the relation of the planar hierarchy to known integrable hierarchies, and in particular to known symmetry reductions of KP/Toda. This gives us the possibility to use various (sometimes in principle non-perturbative) quantization schemes of the tree-level hierarchy, which can be used effectively to reconstruct the higher genus theory [15].

3.2.3. The computational scheme. Let us summarize concretely how we will apply the machinery of Sect. 3.1.2–3.2.2 to solve the topological A-model on a toric open string background (X, L) .

1. Compute the disc factor $D^{X,L}(d, f)$ that specifies the form of the brane insertion operator (3.8). We can do this in two ways:
 - from (3.15), (3.14) using the basic building block (3.9), or
 - using the formula (3.18), and taking the chamber c' to correspond to a suitable orbifold point of the form $[\mathbb{C}^3/G]$.
2. Determine the planar hierarchy of X_T , where the T -action is specified by L and the framing as in (3.1). To do that we need
 - (a) the expression of the tt -metric η , and
 - (b) the expression for the structure constants Γ of $QH_T^\bullet(X)$, both of which are determined by the toric data: η by classical equivariant intersection theory on $X^{\odot T}$, and Γ by the J -function using (A.2), (A.7), (A.9). Multi-pointed amplitudes are computed from the flows (3.23), or from the J -function and the recursion relations (3.33), (3.29).
3. Compute higher genus corrections by
 - using the $3g - 2$ theorem and universal relations (e.g. (3.38), (3.41)), or
 - the quantization formalism of semi-simple Frobenius structures [48], or
 - a full-dispersive formulation of the tree-level hierarchy, as in [15].

We will see in the next section this formalism at work in a number of examples.

4. Examples

4.1. The framed vertex. As a first example of our formalism, let us first consider the framed topological vertex $X = \mathbb{C}^3$ with a toric brane ending on the x_3 leg. This example was already considered in [59] (see also [83, 84] for recent work directly relevant to the BKMP theory); what we add here is a detailed analysis of the dual closed string theory. Equation (3.1) becomes

$$\begin{aligned} (T \simeq \mathbb{C}^*) \times \mathbb{C}^3 &\rightarrow \mathbb{C}^3, \\ (\mu, x_1, x_2, x_3) &\rightarrow (\mu^f x_1, \mu^{-f-1} x_2, \mu x_3). \end{aligned} \quad (4.1)$$

The chiral ring, consisting of the sole identity class, is the trivial algebra structure on the field of fractions $\mathbb{C}(\lambda)$ of $H_T(\text{pt})$,

$$H_T(X) = \text{span}_{\mathbb{C}(\lambda)} \mathbf{1}, \quad (4.2)$$

and the 1×1 tt -metric is

$$\eta(\mathbf{1}, \mathbf{1}) = -\frac{1}{f(f+1)\lambda^3}. \quad (4.3)$$

Following the discussion in Sect. 3.1.2 about λ -homogeneity of open string amplitudes, we will henceforth suppress consistently the λ -dependence everywhere by setting $\lambda = 1$.

4.1.1. The brane insertion operator. The brane insertion operator at framing f is simply given by the specialization of (3.9) to the case $p = 1$,

$$D^{\mathbb{C}^3, L} = \frac{\Gamma(fd+d)}{d!\Gamma(df+1)} \mathbf{1}. \quad (4.4)$$

4.1.2. The genus zero hierarchy and dispersionless KdV. Let us now construct the dispersionless hierarchy governing the T -equivariant tree level theory. Since the chiral ring is a trivial one-dimensional unital algebra, the deformed Gauss-Manin connection on $TH_T(X)$ is simply

$$\nabla_z = \partial_t + \frac{1}{z} \quad (4.5)$$

in an affine chart of $H_T(X)$ parameterized by $t \in \mathbb{C}(\lambda)$. Flat co-ordinates for (4.5) satisfy by definition the ODE

$$z\partial_t^2 h(t, z) = \partial_t h(t, z). \quad (4.6)$$

A family of solutions of (4.6) is given by

$$h(t, z) = A(z)e^{t/z} + B(z). \quad (4.7)$$

The string equation (3.30) and comparison with the twisted J -function (see Appendix A) set

$$A(z) = z, \quad B(z) = -z. \quad (4.8)$$

Our sought-for planar hierarchy is then given by a set of compatible quasi-linear conservation laws on the field space $\mathcal{F} = L(H_T(X))$, endowed with the Poisson bracket (3.24). They take the form

$$\frac{\partial u(x)}{\partial t_{k-1}} := \left\{ u(x), \int_{S^1} \frac{u^{k+1}(y)}{(k+1)!} \right\} = \begin{cases} 0 & k = 0 \\ \frac{u(x)^k u_x(x)}{k!} & k > 0 \end{cases}. \quad (4.9)$$

This hierarchy is the dispersionless limit of the Korteweg-de Vries hierarchy. The t^0 -flow amounts to space translations

$$u_{t^0} = u_x, \quad (4.10)$$

whereas the t^1 -flow

$$u_{t^1} = uu_x \quad (4.11)$$

is given by the zero-dispersion, $\epsilon \rightarrow 0$ limit of the KdV equation $u_{t^1} = uu_x + \epsilon u_{xxx}$. The orbit corresponding to the planar full-descendent potential is cut out by the Kontsevich initial datum

$$u(t^0 = x) \Big|_{t_k=0 \text{ for } k>0} = x, \quad (4.12)$$

and the resulting τ -function is therefore the genus zero limit of the Witten-Kontsevich τ -function [61, 81].

The fact that the closed planar theory reduces to genus zero topological gravity is expected: the genus g full-descendent Gromov-Witten potential of \mathbb{C}^3 is

$$\mathcal{F}_g^{\mathbb{C}^3, T}(t_p, f) = \sum_{n=0}^{\infty} \sum_{p_1, \dots, p_n} \frac{\prod_{i=1}^n t_{p_i}}{n!} \langle \mathcal{O}_{p_1} \dots \mathcal{O}_{p_n} \rangle_{g,n}^{\mathbb{C}_T^3}, \quad (4.13)$$

where

$$\langle \mathcal{O}_{p_1} \dots \mathcal{O}_{p_n} \rangle_{g,n}^{\mathbb{C}_T^3} = \int_{\mathcal{M}_{g,n}} \Lambda_g^\vee(1) \Lambda_g^\vee(f) \Lambda_g^\vee(-f-1) \prod_{i=1}^n \psi_i^{p_i} \quad (4.14)$$

and $\Lambda_g^\vee(x) = \sum_{i=0}^g (-1)^i x^{g-i} \lambda_i^{(g)}$, where $\lambda_i^{(g)} = c_i(\mathbb{E}_g)$ is the i^{th} Chern class of the Hodge bundle on the moduli space of curves; the three Hodge insertions are the normal contribution of each \mathbb{C} -fiber to the A-model on the T -fixed point. At genus zero, though, $\Lambda_0^\vee(x) = 1$ and up to a trivial normalization of the metric (4.3) we boil down to the Witten-Kontsevich case.

The resulting integrable structure (4.9) is a remarkably simple one. It is well-known from the topological vertex formalism that the one-legged framed vertex is governed by a τ -function of the KP-hierarchy [85]; the degree of sophistication only increases when considering two-legged [85] and three-legged setups [2], where the relevant integrable hierarchy coincides respectively with the $2D$ -Toda and the 3-KP hierarchy. Equation (3.7) gives a new perspective in terms of simpler 1+1 (as opposed to 2+1) dimensional integrable systems: the relevant integrable hierarchy is a dispersive deformation of the (simplest) 1+1 dimensional integrable hierarchy, namely the dKdV hierarchy, and as we will see this statement continues to hold when considering multi-legged configurations.

4.1.3. String loops and quasi-Miura triviality. To deform the tree-level hierarchy (4.9), let us apply the machinery of Sect. 3.2.2. We will be looking for a rational Miura transformation of the form

$$u(x, g_s) = u(x)_{[0]} + \frac{\partial^2 (g_s^2 \mathcal{F}([u], g_s) - \mathcal{F}_0(u))}{\partial x^2} \quad (4.15)$$

where, using the $3g - 2$ theorem, we have

$$\mathcal{F}([u], g_s) - \frac{\mathcal{F}_0(u)}{g_s^2} = \mathcal{F}_1(u_x, u) + g_s^2 \mathcal{F}_2(u, u_x, u_{xx}, u_{xxx}, u^{(IV)}) + \dots \quad (4.16)$$

In this language the topological recursion relations become a set of differential identities for the coefficients of the jet variables $u^{(j)}$ in (4.16). For example, at genus 1 (3.38) implies

$$\mathcal{F}_1(u, u_x) = \frac{1}{24} \log u_x + F_1(u)m, \quad (4.17)$$

where the term $F_1(u)$ is the non-descendent genus one free energy. A trivial Hodge-integral computation for the only non-vanishing primary invariant at genus one yields

$$F_1(u) = \alpha(f) \frac{u}{24} \quad (4.18)$$

with

$$\alpha(f) = \frac{f^2 + f + 1}{f(f+1)} \quad (4.19)$$

which fixes (4.17) completely.

At two-loops we can argue similarly and obtain from (3.41),

$$\begin{aligned} \mathcal{F}_2(u, u_{xx}, u_{xxx}, u^{(IV)}) = & \frac{u^{(4)}(x)}{1152u'(x)^2} - \frac{7u^{(3)}(x)u''(x)}{1920u'(x)^3} + \frac{u''(x)^3}{360u'(x)^4} + \frac{7\alpha(f)^2u''(x)}{5760} \\ & - \frac{11\alpha(f)u''(x)^2}{5760u'(x)^2} + \frac{\alpha(f)u^{(3)}(x)}{480u'(x)} + \frac{\beta(f)u'(x)^2}{5760}, \end{aligned} \quad (4.20)$$

where the parameter $\beta(f)$ is fixed by a non-descendent computation on $\overline{\mathcal{M}}_{2,0}$ as

$$\beta(f) = -\frac{1}{f(f+1)}. \quad (4.21)$$

Notice that if we set $\alpha(f)$ and $\beta(f)$ equal to zero, we would be left with the well-known expansion for topological gravity [29, 32] which sends the dispersionless KdV τ -function to the all-genus Witten-Kontsevich τ -function. Turning on $\alpha(f)$ and $\beta(f)$ gives rise to a deformation of the KdV hierarchy, which we can straightforwardly read off by plugging in (4.15) into (3.23). For the first few flows we obtain

$$\frac{\partial u}{\partial t_1} = u_x, \quad (4.22)$$

$$\begin{aligned} \frac{\partial u}{\partial t_2} = & u(x)u'(x) + \frac{1}{12}g_s^2 \left[u^{(3)}(x) + \alpha u'(x)u''(x) \right] + \frac{g_s^4}{720} \\ & \times \left[\alpha \left(u^{(5)}(x) + 5\alpha u^{(3)}(x)u''(x) \right) + \beta u^{(3)}(x)u'(x)^2 \right. \\ & \left. + u'(x) \left(\alpha^2 u^{(4)}(x) + 2\beta u''(x)^2 \right) \right] + O(g_s^6). \end{aligned} \quad (4.23)$$

$$\begin{aligned} \frac{\partial u}{\partial t_3} = & \frac{1}{2}u(x)^2u'(x) + \frac{1}{24}g_s^2 \left(2u(x)u^{(3)}(x) + \alpha u'(x)^3 + 2(\alpha u(x) + 2)u'(x)u''(x) \right) \\ & + \frac{g_s^4}{720} \left[(\alpha u(x) + 3) \left(u^{(5)}(x) + 5\alpha u^{(3)}(x)u''(x) \right) + u^{(3)}(x)u'(x)^2 \left(\beta u(x) + 5\alpha^2 \right) \right. \\ & \left. + 2\beta u'(x)^3u''(x) + u'(x) \left(\alpha u^{(4)}(x)(\alpha u(x) + 8) + 2u''(x)^2 \left(\beta u(x) + 5\alpha^2 \right) \right) \right] \\ & + O(g_s^6). \end{aligned} \quad (4.24)$$

The scalar integrable hierarchy that arises, and whose form is apparently new, is interesting for at least three reasons. First of all it is known that Hodge integrals can be reduced, using Grothendieck-Riemann-Roch and Faber's algorithm [42], to intersection numbers of ψ classes on $\overline{\mathcal{M}}_{g,n}$. From our point of view, (4.17) and (4.20) can be regarded

as a realization of this statement in the language of integrable hierarchies: that is, the Korteweg-de Vries hierarchy and its Hodge-deformation (4.24) are found to be related by a quasi-Miura transformation of the form

$$u_{\text{KdV}}(x) = u_{\mathbb{C}^3}(x) + \frac{\partial^2}{\partial x^2} \left[\frac{1}{24} \alpha g_s^2 u_{\mathbb{C}^3}(x) + \frac{g_s^4}{5760 u'_{\mathbb{C}^3}(x)^2} (-\alpha u''_{\mathbb{C}^3}(x)^2 + \beta u'_{\mathbb{C}^3}(x)^4 + 2\alpha u_{\mathbb{C}^3}^{(3)}(x) u'(x) + 7\alpha^2 u'_{\mathbb{C}^3}(x)^2 u''_{\mathbb{C}^3}(x)) + O(g_s^6) \right]. \quad (4.25)$$

Secondarily, it is worthwhile to point out that a remarkable property of the KdV hierarchy goes through to this deformed case: despite the rather involved *rational* structure of the Miura transformation, at the level of the equations of motion we find that the flows are *polynomial* in the jet variables, with a delicate cancellation of the denominators in the final expressions. It would be interesting to investigate the properties of this scalar hierarchy in more detail; we plan to investigate this in future work.

Finally, it is interesting to remark that this *very same* hierarchy governs the framed topological vertex with more complicated brane setups, such as the 3-legged vertex. Indeed, in this case the dual closed string theory computes multi-partitions cubic Hodge integrals [27, 67]; at the level of Eqs. (3.7), (4.4), (4.14), and (4.17)–(4.20), this *only amounts* to replace the framing dependent co-efficients $\alpha(f)$, $\beta(f)$ with $\alpha(f_1, \dots, f_l)$, $\beta(f_1, \dots, f_l)$, which can be computed exactly as before starting by replacing (4.14) with

$$\int_{\mathcal{M}_{g, n_1+n_2+n_3}} \Lambda_g^\vee(\rho_1) \Lambda_g^\vee(\rho_2) \Lambda_g^\vee(\rho_3) \prod_{i=1}^n \psi_i^{p_i} \quad (4.26)$$

with $f_i = \rho_{i+1}/\rho_i$; apart from this modification, the underlying integrable hierarchy, will still have the form (4.24). Computing l -legged string amplitudes - that is, multi-trace correlation functions in a matrix model language - corresponds to act with the loop insertions operator, having the same form (4.4), and carrying a different framing parameter for each leg. For example, for the 2- and 3-legged vertex we find

$$\begin{aligned} \alpha(\rho_1, \rho_2) &= \frac{\rho_1^2 + \rho_1 \rho_2 + \rho_2^2}{\rho_1 (\rho_1 \rho_2 + \rho_2^2)}, & \beta(\rho_1, \rho_2) &= -\frac{1}{\rho_1^2 \rho_2 + \rho_1 \rho_2^2}, \\ \alpha(\rho_1, \rho_2, \rho_3) &= \frac{\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1}{\rho_1 \rho_2 \rho_3}, & \beta(\rho_1, \rho_2, \rho_3) &= \frac{1}{\rho_1 \rho_2 \rho_3}. \end{aligned} \quad (4.27)$$

4.1.4. Framed open string amplitudes. With (4.4) and (4.17), (4.20) at hand it is straightforward to compute open string amplitudes for the framed vertex from (3.7). Up to the sign ambiguities affecting open string invariants, we recover the known results for the topological string on \mathbb{C}^3 , with the coefficients of the w -expansion expressed in closed form in the winding number d and the framing f :

$$F_{0,1}^{X,L}(f, w) = \sum_{d=1}^{\infty} \frac{\Gamma(fd + d)}{dd! \Gamma(df + 1)} w^d, \quad (4.28)$$

$$F_{0,2}^{X,L}(f, w_1, w_2) = \sum_{d_1, d_2} \frac{f(f+1) \Gamma(fd_1 + d_1) \Gamma(fd_2 + d_2)}{(d_1 - 1)! (d_2 - 1)! (d_1 + d_2) \Gamma(d_1 f + 1) \Gamma(d_2 f + 1)} w_1^{d_1} w_2^{d_2}, \quad (4.29)$$

$$F_{1,1}^{X,L}(f, w) = \sum_{d=1}^{\infty} \frac{(1 - (d-1)f(f+1))\Gamma(fd+d)}{24(d-1)!\Gamma(df+1)} w^d, \quad (4.30)$$

$$F_{1,2}^{X,L}(f, w_1, w_2) = \sum_{d_1, d_2} \frac{\Gamma(fd_1+d_1)\Gamma(fd_2+d_2)}{24d_1!d_2!\Gamma(d_1f+1)\Gamma(d_2f+1)} w_1^{d_1} w_2^{d_2} \left[d_1^2(-f)(f+1) \right. \\ \left. + d_1(1 - (d_2-1)f(f+1)) - (d_2-1)d_2f(f+1) + d_2 \right], \quad (4.31)$$

$$F_{2,1}^{X,L}(f, w) = \sum_{d=1}^{\infty} \frac{d\Gamma(fd+d)}{5760(d-1)!\Gamma(df+1)} \left[(5d^3f^2(f+1)^2 - 12d^2f(f+1) \right. \\ \left. \times (f^2+f+1) + 7d(f^2+f+1)^2 - 2f(f+1)) \right] w^d, \quad (4.32)$$

$$F_{2,2}^{X,L}(f, w_1, w_2) = \sum_{d_1, d_2} \frac{\Gamma(fd_1+d_1)\Gamma(fd_2+d_2)w_1^{d_1}w_2^{d_2}}{5760d_1!d_2!\Gamma(d_1f+1)\Gamma(d_2f+1)} \left[5d_1^5f^2(f+1)^2 \right. \\ + 3d_1^4f(f+1)((5d_2-4)f(f+1)-4) + d_1^3((d_2-1) \\ \times f(f+1)-1)((29d_2-7)f(f+1)-7) + d_1^2(29d_2^3f^2(f+1)^2 \\ - 50d_2^2f(f+1)(f^2+f+1) + 21d_2(f^2+f+1)^2 \\ - 2f(f+1)) + 3d_1d_2(5d_2^3f^2(f+1)^2 - 12d_2^2f(f+1) \\ \times (f^2+f+1) + 7d_2(f^2+f+1)^2 - 2f(f+1)) \\ + d_2^2(5d_2^3f^2(f+1)^2 - 12d_2^2f(f+1) \\ \times (f^2+f+1) + 7d_2(f^2+f+1)^2 - 2f(f+1)) \left. \right]. \quad (4.33)$$

4.2. The resolved conifold. As a further example, consider the resolved conifold geometry $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with a brane on an outer leg. To simplify our formulae and to make contact more easily with the results of [15], we will restrict the discussion here to framing one, with the generalization to arbitrary framing being completely straightforward.

4.2.1. Geometry and phase space data. The toric diagram of X is depicted in Fig. 1; the skeleton of its fan is given by the 1-dimensional rays generated by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (4.34)$$

We denote by x_i the homogeneous co-ordinate associated to v_i . The GKZ-extended Kähler moduli space $\overline{\mathcal{M}}_X$ of X is isomorphic to \mathbb{P}^1 (Fig. 7); it has two boundary divisors

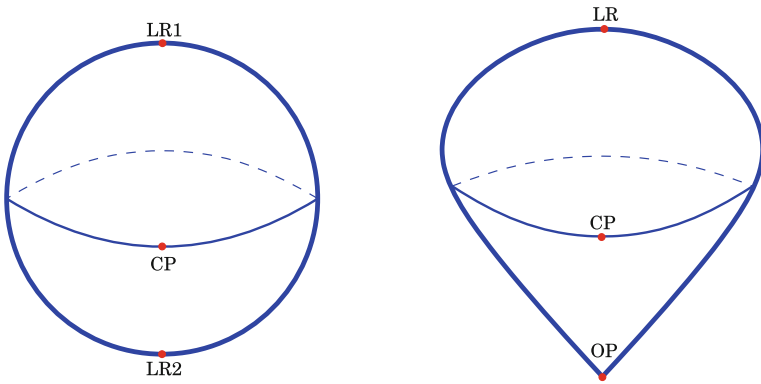


Fig. 7. A pictorial view of the stringy Kähler moduli spaces of the resolved conifold (left) and local \mathbb{P}^2 (right). The red dots denote the large radius, orbifold and conifold divisors in each case

associated to the large volume limit, which are related to each other by a toric flop, and a conifold point. In the “north” (resp. “south”) patch of \mathbb{P}^1 , X is described as a holomorphic quotient

$$X = \frac{\mathbb{C}^4 \setminus Z_X}{(\mathbb{C}^*)}, \quad (4.35)$$

where $Z_X = (0, 0, x_3, x_4)$ (resp. $Z_X = (x_1, x_2, 0, 0)$) and we quotient by a \mathbb{C}^* action with weights $(1, 1, -1, -1)$. Topological string amplitudes are flop-invariant; focusing on the “north” patch, the x_1 and x_2 variables are homogeneous co-ordinates for the null section $X_0 \simeq \mathbb{P}^1 \hookrightarrow X$, and x_3 and x_4 are fiber co-ordinates. The T -action in GLSM co-ordinates reads

$$\begin{array}{ccc} T \times & X & \rightarrow X \\ \mu, & (x_1, x_2, x_3, x_4) & \rightarrow (x_1, x_2, \mu x_3, \mu^{-1} x_4) \end{array} \quad (4.36)$$

that is, it covers the trivial action on the base \mathbb{P}^1 and rotates the fibers *anti*-diagonally.

The T -equivariant chiral ring localizes classically on the co-homology of \mathbb{P}^1 . Denote by $\phi_1 := \mathbf{1} = [\mathbb{P}^1]^\vee \in H^0(X, \mathbb{C})$, $\phi_2 := [\text{pt}]^\vee \in H^2(X, \mathbb{C})$ the identity and the Kähler class respectively. As a vector space,

$$H_T(X) = \text{span}_{\mathbb{C}(\lambda)} \{\phi_1, \phi_2\} \quad (4.37)$$

endowed with a tt -metric given by Atiyah-Bott localization as

$$\eta_{ij} = -\delta_{i+j,3}\lambda. \quad (4.38)$$

For $\phi \in H_T(X)$, write $\phi = t^1 \phi_1 + t^2 \phi_2$. We find from (4.34) and (A.7) that the J -function of X with torus action (4.36) is

$$J_X(t^1, t^2, z) = e^{t^1 \phi_1 / z + t^2 \phi_2 / z} \sum_{d=0}^{\infty} \frac{\prod_{-d+1}^0 (-\phi_2 + mz + \lambda)(-\phi_2 + mz - \lambda)}{\prod_1^d (\phi_2 + mz)^2} e^{dt^2}. \quad (4.39)$$

As before we suppress the λ dependence from now on.

4.2.2. The brane insertion operator. The brane insertion operator has a particularly simple form in this case. In the limit of canonical framing, which amounts to switching off the torus action on the base \mathbb{P}^1 , (3.14) becomes simply

$$D_{\alpha}^{X,L}(d) = \frac{1}{d} \delta_{\alpha,1}. \quad (4.40)$$

4.2.3. The genus zero hierarchy and dispersionless Ablowitz-Ladik. The planar hierarchy has again a very explicit construction, which was discussed in detail in [15]. The Christoffel symbols of the deformed Gauss-Manin connection (3.26) can be read off from the large z asymptotics of $J(z)$,

$$J_{\alpha}(z) = z + t_{\alpha} + \frac{\partial_{\alpha} F_0}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad (4.41)$$

and we find

$$\Gamma_{1i}^j = \delta_i^j, \quad \Gamma_{22}^1 = \frac{e^{t^2}}{e^{t^2} - 1}, \quad \Gamma_{22}^2 = 0. \quad (4.42)$$

Flat co-ordinates for ∇_z satisfy

$$z \partial_i \partial_j h^{\alpha}(z) = \Gamma_{ij}^k \partial_k h^{\alpha}(z). \quad (4.43)$$

By (4.42), the system of PDEs reduces to solve ODEs for $i = j = 1$, $i = 1$ and $j = 2$, and finally $i = j = 2$. Comparison with the J -function of X equivariant w.r.t. (4.36) (see [15] for the complete calculation) yields

$$h^1(t^1, t^2, z) = -h_2(t^1, t^2, z) = z \left(e^{t^1/z} {}_2F_1 \left(-1/z, 1/z; 1; e^{t^2} \right) - 1 \right), \quad (4.44)$$

$$\begin{aligned} h^2(t^1, t^2, z) = -h_1(t^1, t^2, z) = e^{t^1/z} & \left[-1/z \left(\psi^{(0)}(1 + 1/z) \right. \right. \\ & \left. \left. + \psi^{(0)}(1 - 1/z) + 2\gamma \right) {}_2F_1 \left(1/z, -1/z; 1; e^{t^2} \right) \right. \\ & \left. - \frac{\pi \left(1 - e^{t^2} \right)}{z^2 \sin(\pi/z)} {}_2F_1 \left(1 + 1/z, 1 - 1/z; 2; 1 - e^{t^2} \right) \right], \end{aligned} \quad (4.45)$$

where ${}_2F_1(a, b; c; x)$ denotes Gauss' hypergeometric function, $\psi^{(0)}(z) = \frac{d \log \Gamma(z)}{dz}$ is the digamma function, and γ is the Euler-Mascheroni constant.

As pointed out in [15], the resulting hierarchy is the continuum limit of a 2-component integrable lattice: the Ablowitz-Ladik hierarchy [1]. For the first few flows we have

$$\frac{\partial u^1}{\partial t_{1,0}} = u_x^1, \quad (4.46)$$

$$\frac{\partial u^2}{\partial t_{1,0}} = u_x^2, \quad (4.47)$$

$$\frac{\partial u^1}{\partial t_{2,0}} = -\frac{e^{u^2}}{1 - e^{u^2}} u_x^2, \quad (4.48)$$

$$\frac{\partial u^2}{\partial t_{2,0}} = u_x^1. \quad (4.49)$$

By (4.40), open string amplitudes are controlled in the dual theory by gravitational descendents of the Kähler class¹⁰ ϕ_2 , which are in turn associated to the $t_{2,p}$ flows of (4.44). At genus zero, amplitudes with an arbitrary number of holes are then completely determined by (4.44), (3.33) and (3.29). We find for example, denoting as usual $t := t^2$ the Kähler volume of the base \mathbb{P}^1 ,

$$F_{0,1}^{X,L}(t, w, f) = \sum_{d=1}^{\infty} {}_2F_1(d, -d; 1; e^t) \frac{w^d}{d^2}, \quad (4.50)$$

$$F_{0,2}^{X,L}(t, w, f) = \sum_{d_1, d_2} \frac{w_1^{d_1} w_2^{d_2} e^t}{d_1 + d_2} \left[d_2 {}_2F_1(-d_1, d_1; 1; e^t) {}_2F_1(1 - d_2, d_2 + 1; 2; e^t) \right. \\ \left. + d_1 {}_2F_1(1 - d_1, d_1 + 1; 2; e^t) {}_2F_1(-d_2, d_2; 1; e^t) \right] \quad (4.51)$$

in perfect agreement with the known results.

4.2.4. The higher genus theory. We have two ways to add loop corrections: we could either deform the planar hierarchy by using the universal identities (3.38)–(3.41) or by exploiting knowledge of the full dispersive hierarchy [1]. Either way, we find for example that the generating function of the rational Miura transformation at one loop reads

$$\mathcal{F}_1^X(u^2, u_x^1, u_x^2) = \frac{1}{24} \log \left(u_x^1(x)^2 + \frac{\lambda^2 e^{u^2(x)}}{1 - e^{u^2(x)}} u_x^2(x)^2 \right) + \frac{1}{12} \text{Li}_1(e^{u^2(x)}) + \frac{u^2(x)}{24}. \quad (4.52)$$

We obtain

$$F_{1,1}^{X,L}(t, w) \\ = \sum_{d=1}^{\infty} \frac{2d^2 e^t (e^t - 1) {}_2F_1(1 - d, d + 1; 2; e^t) + (3e^t - 1) {}_2F_1(-d, d; 1; e^t)}{24(e^t - 1)} w^d, \quad (4.53)$$

and at two-loops

$$F_{2,1}^{X,L}(t, w) = \sum_{d=1}^{\infty} \left[\left(-20d^4 e^t (e^t - 1)^2 + 24d^3 (e^t - 1)^2 (3e^t - 1) - d^2 (e^t - 1) \right. \right. \\ \times (e^t (63e^t + 2) + 7) + 4d (e^t - 1) (10e^t + e^{2t} + 1) - 24e^t (e^t + 1) \Big) \\ \times {}_2F_1(-d, d; 1; e^t) - 4d (e^t - 1) (6d^2 (-4e^t + 3e^{2t} + 1) \\ \left. + 10e^t + e^{2t} + 1) {}_2F_1(1 - d, d; 1; e^t) \right] \frac{w^d}{5760(e^t - 1)^3}, \quad (4.54)$$

in complete agreement with the Wilson-loop computation for the unknot in Chern-Simons theory.

¹⁰ The fact that the open string partition function is insensitive to the descendents of the unit class is somewhat reminiscent of the \mathbb{P}^1 topological string, where this type of insertions are completely invisible both in the mirror symmetry description of non-normalizable modes [2] and in its gauge theory realization as a deformed $U(1) \mathcal{N} = 2$ Yang-Mills theory in four-dimensions [72].

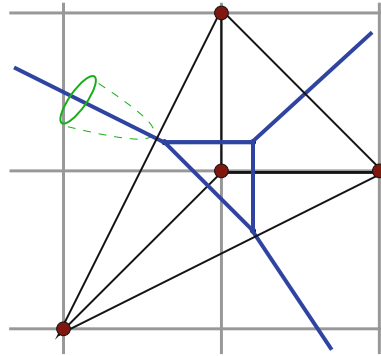


Fig. 8. The toric diagram (black) and the web diagram (blue) of the local \mathbb{P}^2 geometry with a brane on an outer leg

4.3. Local \mathbb{P}^2 . Up to this point, we have considered Calabi-Yau geometries whose mirrors are controlled by genus zero spectral curves. We move here to the case of local surfaces, for which the mirror geometry is encoded in a family of elliptic curves. The underlying integrable structure is more difficult to describe in this case; we will see how our formalism goes through.

As a concrete example, we take X to be the total space $K_{\mathbb{P}^2}$ of the canonical line bundle over the complex projective plane. In order to illustrate the discussion of Sect. 3.1.4 in this example, we consider a configuration given by a toric brane on an outer leg (see Fig. 8) at generic framing.

4.3.1. Geometry and phase space data. The toric diagram of X is depicted in Fig. 8; the skeleton of its fan is given by the 1-dimensional rays generated by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \quad (4.55)$$

We again denote by x_i the homogeneous co-ordinate associated to v_i . The stringy Kähler moduli space $\overline{\mathcal{M}}_X$ of X is isomorphic to $\mathbb{P}^{(1,3)}$ (Fig. 7); it has one boundary divisor associated to the large volume limit, a conifold point, and a \mathbb{Z}_3 -orbifold point, corresponding to the classical tip of the Kähler cone where the \mathbb{P}^2 divisor shrinks to zero volume. In the large radius patch, X is described as the holomorphic quotient

$$X = \frac{\mathbb{C}^4 \setminus Z_X}{(\mathbb{C}^*)}, \quad (4.56)$$

where now $Z_X = (0, 0, 0, x_4)$ and we quotient by a \mathbb{C}^* action with weights $(1, 1, 1, -3)$; x_i , $i = 1, 2, 3$ give homogeneous co-ordinates for $\{x_4 = 0\} \simeq \mathbb{P}^2 \hookrightarrow X$, and x_4 coordinatizes the \mathbb{C} -fiber. An “outer” brane in this setup intersects the equator of the fiber; we can take the T -action to be

$$\begin{array}{ccc} T \times & X & \rightarrow \\ \mu, & (x_1, x_2, x_3, x_4) & \rightarrow (x_1, \mu^f x_2, \mu^{-f-1} x_3, \mu x_4). \end{array} \quad (4.57)$$

Any permutation of (x_1, x_2, x_3) would yield the same result; this reflects the symmetry of the three outer legs in Fig. 8.

Let $p \in H^2(\mathbb{P}^2, \mathbb{C})$ denote the hyperplane class $c_1(\mathcal{O}_{\mathbb{P}^2}(1))$. We still denote by p its lift to the T -equivariant co-homology of X ,

$$H_T(X) = \frac{\mathbb{C}(\lambda)[p]}{\langle p^3 = \lambda p^2 + f(f+1)\lambda^2 p \rangle}. \quad (4.58)$$

The tt -pairing $(\phi_{(1)}, \phi_{(2)}) = \sum_{i,j} \eta_{ij} c_{(1)}^i c_{(2)}^j$ between elements $\phi^{(a)} = \sum_{i=0}^2 c_{(a)}^i p^i$ of the chiral ring is specified by the Gram matrix in the p^i basis as

$$\eta_{ij} = \int_{X \circ T} p^{i+j} = \sum_{m=1}^3 \text{Res}_{p=p_m} \frac{p^{i+j}}{p(p+f\lambda)(p-(f+1)\lambda)(-3p+\lambda)}, \quad (4.59)$$

where $p_1 = 0$, $p_2 = -f\lambda$, $p_3 = (f+1)\lambda$.

Some of our computations will be more naturally expressed in classical canonical co-ordinates - that is, in the basis of idempotents $\{\zeta_1, \zeta_2, \zeta_3\}$ of classical T -equivariant co-homology. It is straightforward to check that

$$\begin{aligned} \zeta_1 &= 1 + \frac{1}{f+f^2} \left(\frac{p}{\lambda} - \frac{p^2}{\lambda^2} \right), \\ \zeta_2 &= \frac{1}{1+3f+2f^2} \left(\frac{fp}{\lambda} + \frac{p^2}{\lambda^2} \right), \\ \zeta_3 &= \frac{1}{f+2f^2} \left(\frac{p^2}{\lambda^2} - \frac{(1+f)p}{\lambda} \right). \end{aligned} \quad (4.60)$$

At the \mathbb{Z}_3 -orbifold point we have $Z_X = (x_1, x_2, x_3, 0)$; the resulting toric variety is the coarse moduli space $\mathbb{C}^3/\mathbb{Z}_3$ of the orbifold $\mathcal{X} := [\mathbb{C}^3/\mathbb{Z}_3]$, where the cyclic group acts diagonally with unit weight on $\mathbb{C}^3 \simeq \{(w^{1/3}x_1, w^{1/3}x_2, w^{1/3}x_3)\}$. The T -action descends on $\mathbb{C}^3/\mathbb{Z}_3$ as

$$\begin{array}{ccc} T \times & \mathbb{C}^3/\mathbb{Z}_3 & \rightarrow \\ \mu, (w^{1/3}x_1, w^{1/3}x_2, w^{1/3}x_3) & \rightarrow & (\mu^{1/3}w^{1/3}x_1, w^{1/3+f}x_2, w^{1/3-f-1}x_3). \end{array} \quad (4.61)$$

At the orbifold point, the classical chiral ring $H_T^{\text{orb}}(\mathcal{X}) = \bigoplus_{k=1}^3 H_T(\mathcal{X}_k)$ is generated by twisted classes $\mathbf{1}_{\frac{k}{3}}$, $k = 0, 1, 2$, where \mathcal{X}_k denotes the k^{th} twisted sector of \mathcal{X} . Write $\psi = \sum_{k=0}^2 d^k \mathbf{1}_{\frac{k}{3}}$ for $\psi \in H_T^{\text{orb}}(\mathcal{X})$. The tt -metric at this point is just the orbifold Poincaré pairing

$$\eta_{kl}^{\text{orb}} = -\frac{9\delta_{k0}\delta_{l0}}{f(f+1)\lambda^3} + \frac{\delta_{k+l,3}}{3}. \quad (4.62)$$

4.3.2. The brane insertion operator. We set $\lambda = 1$ as before. We want to compute open topological string amplitudes for the configuration at hand at the orbifold and at the large radius point. To do that, we will exploit the point of view of Sect. 3.1.4: we use the localization formula at the orbifold point and impose the invariance condition

$$D^{\text{LR}}(d, f) \cdot I^{(X)}\left(t; \frac{1}{d}, f\right) = D^{\text{orb}}(d, f) \cdot I^{(\mathcal{X})}\left(t^{\text{orb}}; \frac{1}{d}, f\right), \quad (4.63)$$

where t and t^{orb} are small quantum co-homology parameters.

So let us start from the orbifold point. We have from (3.9),

$$D^{\text{orb}}(d, f) = \frac{1}{\lfloor \frac{d}{3} \rfloor!} \left(\frac{1}{d}\right)^{3\langle d/3 \rangle} \frac{\Gamma(\frac{d}{3} + \langle \frac{d}{3} \rangle + d(f + \frac{1}{3}))}{\Gamma(1 - \langle \frac{d}{3} \rangle + d(f + \frac{1}{3}))} \mathbf{1}_{3\langle d/3 \rangle}. \quad (4.64)$$

To compute the large radius loop insertion operator we employ (4.63) and the chamber-crossing formulas (3.17), (3.18). To compute $M^{\text{orb,LR}}(z)$, we analytically continue the large radius I -function to the orbifold point, following [22]. At the orbifold point, the I -function of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$ reads from (A.9),

$$I_{\mathcal{X}}(x, f, z) = z x^{-1/z} \sum_{l \geq 0} \frac{x^l}{l! z^l} \prod_{\substack{b: 0 \leq b < \frac{l}{3} \\ \langle b \rangle = \lfloor \frac{l}{3} \rfloor}} \left(\frac{1}{3} - bz\right) \left(\frac{1}{3} + f - bz\right) \left(\frac{1}{3} - f - 1 - bz\right) \mathbf{1}_{\lfloor \frac{l}{3} \rfloor}. \quad (4.65)$$

The large z asymptotics fixes

$$J_{\mathcal{X}}(t_{\text{orb}}, f, z) = x^{1/z} I_{\mathcal{X}}(x(t_{\text{orb}}), f, z), \quad (4.66)$$

where

$$t_{\text{orb}}(x) = \sum_{m \geq 0} (-1)^m \frac{x^{3m+1}}{(3m+1)!} \frac{\Gamma(m + \frac{1}{3})^3}{\Gamma(\frac{1}{3})^3}. \quad (4.67)$$

At large radius, we have from (A.7) with $Y = \mathbb{P}^2$, $X = K_{\mathbb{P}^2}$ and the torus action (4.57) that

$$I_X(y, f, z) = z \sum_{d \geq 0} \frac{\prod_{-3d < m \leq 0} (1 - 3p + mz)}{\prod_{0 < m \leq d} (p + mz)(p + f + mz)(p - f - 1 + mz)} y^{d+p/z}. \quad (4.68)$$

In this case the large z -asymptotics gives for the J -function

$$J_X(t, f, z) = e^{f(y)/z} I_X(y, f, z), \quad (4.69)$$

where

$$e^t = y \exp(3f(y)), \quad f(y) = \sum_{d > 0} \frac{(3d-1)!}{(d!)^3} (-y)^d. \quad (4.70)$$

The power series expansions (4.65) and (4.68) have respectively radius of convergence $|x| < 3$, $|y| < \frac{1}{27}$, where the B-model variables x and y are local co-ordinates around the orbifold and the large radius point respectively. Their relation can be read off from the secondary fan of X ([26]; see Fig. 7) to be $y = x^{-3}$.

We follow closely here [22], with minor modifications due to the effectiveness of the torus action on the base of $K_{\mathbb{P}^2}$. For the purposes of analytic continuation, it will be worthwhile to rewrite the summands in (4.65) and (4.68) in terms of ratios of Γ -functions,

$$I_X(y, f, z) = z \sum_{d \geq 0} \frac{\Gamma(1 + \frac{p}{z}) \Gamma(1 + \frac{p+f}{z}) \Gamma(1 + \frac{p-f-1}{z}) \Gamma(1 + \frac{1-3p}{z})}{\Gamma(1 + \frac{p}{z} + d) \Gamma(1 + \frac{p+f}{z} + d) \Gamma(1 + \frac{p-f-1}{z} + d) \Gamma(1 + \frac{1-3p}{z} - 3d)} y^{d+p/z}. \quad (4.71)$$

An efficient way to compute the analytic continuation of $I_X(y, f, z)$ is to use the Mellin–Barnes method. We first apply Euler’s identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ to (4.71) until each factor $\Gamma(a+bd)$ in the summand has $b > 0$:

$$I_X(y, f, z) = -\Theta_X \sum_{d \geq 0} \frac{\Gamma(3d - \frac{1-3p}{z})}{\Gamma(1 + \frac{p}{z} + d) \Gamma(1 + \frac{p+f}{z} + d) \Gamma(1 + \frac{p-f-1}{z} + d)} (-1)^d y^{d+p/z} \quad (4.72)$$

with

$$\begin{aligned} \Theta_X &= \pi^{-1} z \Gamma\left(1 + \frac{p}{z}\right) \Gamma\left(1 + \frac{p+f}{z}\right) \Gamma\left(1 + \frac{p-f-1}{z}\right) \\ &\quad \times \Gamma\left(1 + \frac{1-3p}{z}\right) \sin\left(\pi \left[\frac{1-3p}{z}\right]\right). \end{aligned} \quad (4.73)$$

Consider now the contour integral in the complex s -plane as depicted in Fig. 9

$$\int_C \Theta_X \frac{\Gamma(3s - \frac{1-3p}{z}) \Gamma(s) \Gamma(1-s)}{\Gamma(1 + \frac{p}{z} + s) \Gamma(1 + \frac{p+f}{z} + s) \Gamma(1 + \frac{p-f-1}{z} + s)} y^{s+p/z}. \quad (4.74)$$

The integral (4.74) is defined and analytic throughout the region $|\arg(y)| < \pi$. For $|y| < \frac{1}{27}$ we close the contour to the right and pick up the residues at $s = n, n \in \mathbb{Z}_+$: this gives us back the large radius expansion (4.72). For $|y| > \frac{1}{27}$ we close the contour to the left, and then (4.74) is equal to the sum of residues at

$$s = -1 - n, \quad n \geq 0, \quad \text{and} \quad s = \frac{1-3p}{3z} - \frac{n}{3}, \quad n \geq 0.$$

The residues at $s = -1 - n, n \geq 0$, vanish in $H(Y)$ as they are divisible by $p^3 - p(f + f^2 + p)$. Thus the analytic continuation \tilde{I}_X of I_X is equal to the sum of the remaining residues:

$$\Theta_X \sum_{n \geq 0} \frac{(-1)^n}{3n!} \frac{\Gamma(\frac{1-3p}{3z} - \frac{n}{3}) \Gamma(1 - \frac{1-3p}{3z} + \frac{n}{3})}{\Gamma(1 + \frac{1}{3} - \frac{n}{3}) \Gamma(1 + \frac{1}{3} - \frac{n}{3} + \frac{f}{z}) \Gamma(1 + \frac{1}{3} - \frac{n}{3} - \frac{f+1}{z})} y^{1/3z-n/3}.$$

Writing this in terms of the co-ordinate $x = y^{-1/3}$, we find that the analytic continuation $\tilde{I}_Y(x, z)$ is equal to

$$z x^{-1/z} \sum_{n \geq 0} \frac{(-x)^n}{3.n!} \frac{\Gamma(\frac{1-3p}{3z} - \frac{n}{3}) \Gamma(1 - \frac{1-3p}{3z} + \frac{n}{3})}{\Gamma(1 + \frac{1}{3} - \frac{n}{3}) \Gamma(1 + \frac{1}{3} - \frac{n}{3} + \frac{f}{z}) \Gamma(1 + \frac{1}{3} - \frac{n}{3} - \frac{f+1}{z})}. \quad (4.75)$$

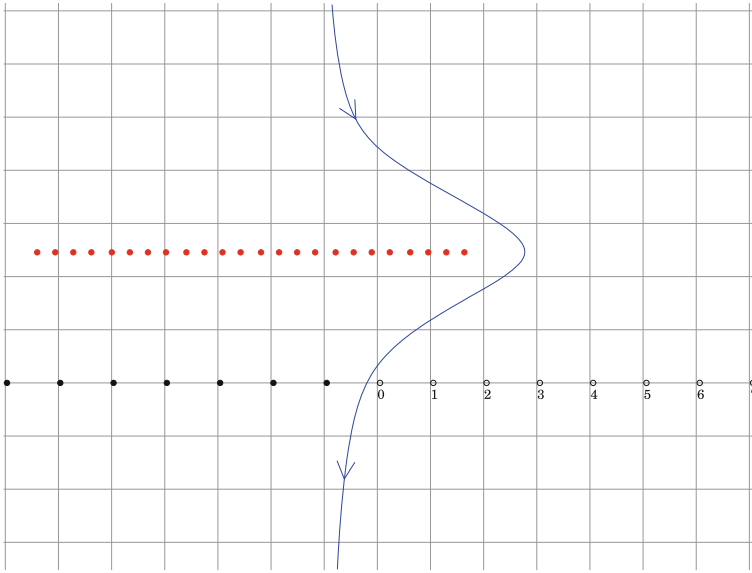


Fig. 9. The contour of integration C in (4.74)

To compute the linear transformation $M : H(X) \rightarrow H(Y)$ that sends the orbifold I-function $I_{\mathcal{X}}$ to \tilde{I}_X , it is sufficient to expand in x the equality $M(z, f)I_{\mathcal{X}}(x, f, z) = \tilde{I}_Y(x, f, z)$. We have

$$I_{\mathcal{X}}(x, z) = z x^{-1/z} \left(\mathbf{1}_0 + \frac{x}{z} \mathbf{1}_{1/3} + \frac{x^2}{2z^2} \mathbf{1}_{2/3} + O(x^3) \right), \quad (4.76)$$

and we obtain from (4.75) that

$$M(\mathbf{1}_0) = \frac{1}{3} \frac{\sin\left(\pi\left[\frac{1-3p}{z}\right]\right)}{\sin\left(\pi\left[\frac{1-3p}{3z}\right]\right)} \frac{\Gamma\left(1+\frac{p}{z}\right)\Gamma\left(1+\frac{p+f}{z}\right)\Gamma\left(1+\frac{p-f-1}{z}\right)\Gamma\left(1+\frac{1-3p}{z}\right)}{\Gamma\left(1+\frac{1}{3z}\right)\Gamma\left(1+\frac{1}{3z}+\frac{f}{z}\right)\Gamma\left(1+\frac{1}{3z}-\frac{f+1}{z}\right)}, \quad (4.77)$$

$$M(\mathbf{1}_{1/3}) = \frac{z}{3} \frac{\sin\left(\pi\left[\frac{1-3p}{z}\right]\right)}{\sin\left(\pi\left[\frac{1-3p}{3z}-\frac{1}{3}\right]\right)} \frac{\Gamma\left(1+\frac{p}{z}\right)\Gamma\left(1+\frac{p+f}{z}\right)\Gamma\left(1+\frac{p-f-1}{z}\right)\Gamma\left(1+\frac{1-3p}{z}\right)}{\Gamma\left(1+\frac{1}{3z}+\frac{1}{3}\right)\Gamma\left(1+\frac{1}{3z}+\frac{f}{z}+\frac{1}{3}\right)\Gamma\left(1+\frac{1}{3z}-\frac{f+1}{z}+\frac{1}{3}\right)}, \quad (4.78)$$

$$M(\mathbf{1}_{2/3}) = \frac{z^2}{3} \frac{\sin\left(\pi\left[\frac{1-3p}{z}\right]\right)}{\sin\left(\pi\left[\frac{1-3p}{3z}-\frac{2}{3}\right]\right)} \frac{\Gamma\left(1+\frac{p}{z}\right)\Gamma\left(1+\frac{p+f}{z}\right)\Gamma\left(1+\frac{p-f-1}{z}\right)\Gamma\left(1+\frac{1-3p}{z}\right)}{\Gamma\left(1+\frac{1}{3z}+\frac{2}{3}\right)\Gamma\left(1+\frac{1}{3z}+\frac{f}{z}+\frac{2}{3}\right)\Gamma\left(1+\frac{1}{3z}-\frac{f+1}{z}+\frac{2}{3}\right)}. \quad (4.79)$$

To determine the change (3.18) of the brane insertion operator, it is convenient to express the images of the isomorphism induced by M in the basis $\zeta_1, \zeta_2, \zeta_3$ of the classical idempotents (4.60) of $H_T(K_{\mathbb{P}^2})$. In this basis, the matrix $M = (m_{ij})$ has the form

$$M_{ij}(z) = M(\mathbf{1}_{(j-1)/3}) \Big|_{p=(1+f)\delta_{i2}-f\delta_{i3}}. \quad (4.80)$$

By the degree-twisting condition for the disc function, we have that

$$\left[M^{-1} \left(d^{-1} \right) \right]^T \cdot D^{\text{orb}}(d, f) = \sum_{i=1}^3 \left[M^{-1} \left(d^{-1} \right) \right]_{i, 3\langle d/3 \rangle + 1}^T D_{3\langle d/3 \rangle}^{\text{orb}}(d, f) \zeta_i. \quad (4.81)$$

For the first few values of d we obtain

$$\begin{aligned} (M^{-1})^T(1)_{i,2} &= \delta_{i1}, \\ (M^{-1})^T(1/2)_{i,3} &= 2\delta_{i1}, \\ (M^{-1})^T(1/3)_{i,1} &= -\frac{1}{6}(1+3f)(2+3f)\delta_{i1}, \\ (M^{-1})^T(1/4)_{i,2} &= -\frac{1}{6}(4f+1)(4f+3)\delta_{i1}, \end{aligned} \quad (4.82)$$

which together with (4.64) and (4.63) yield for the first few values of d ,

$$D^{\text{LR}}(d, f) = D_1^{\text{LR}}(d, f) \zeta_1 \quad (4.83)$$

with

$$\begin{aligned} D_1^{\text{LR}}(1, f) &= 1, \quad D_1^{\text{LR}}(2, f) = -\frac{1+2f}{2}, \\ D_1^{\text{LR}}(3, f) &= \frac{3}{2} \left(-f - \frac{2}{3} \right) \left(f + \frac{1}{3} \right), \dots \end{aligned} \quad (4.84)$$

in complete agreement with (3.14).

4.3.3. Open string phase transitions. As we emphasized in Sect. 3.1.4, the fact that the I-function is a globally defined holomorphic function on the stringy moduli space results in a relative, closed moduli-dependent normalization of the J-functions at the orbifold and the large radius point. In particular from (4.68), (4.65) we have

$$J^X(t, z) = e^{f(y(t))/(3z)} I^X(y(t), z), \quad (4.85)$$

$$J^{\mathcal{X}}(t, z) = y(t)^{1/(3z)} I^{\mathcal{X}}(y(t), z), \quad (4.86)$$

where

$$f(y) = 3 \sum_{d=0}^{\infty} \frac{(3d-1)!}{(d!)^3} (-y)^d \quad (4.87)$$

is the worldsheet instanton correction to the closed mirror map. By (3.13) and (3.20) this means that the A-model flat *open* string moduli are related by a renormalization of the form

$$w_{\text{LR}} = Q^{1/3} w_{\text{orb}}, \quad (4.88)$$

where Q is the exponentiated Kähler parameter; equivalently, in terms of the B-model open modulus $w_{\text{bare}} = e^{-f(y(t))/(3z)} w_{\text{LR}} = y(t)^{-1/(3z)} w_{\text{orb}}$, we have the open mirror maps

$$\ln w_{\text{LR}} = \ln w_{\text{bare}} + \frac{t - \ln y}{3}, \quad (4.89)$$

$$\ln w_{\text{orb}} = \ln w_{\text{bare}} - \frac{\ln y}{3}. \quad (4.90)$$

This is precisely the form of the open mirror map for $K_{\mathbb{P}^2}$ and $[\mathbb{C}^3/\mathbb{Z}_3]$ that we would obtain from the open string Picard Fuchs system [5, 13, 66], with the correct choice of solution automatically picked up at both boundary points.

4.3.4. Computations. Having obtained the expression of the brane insertion operators at the orbifold and the large radius point, we turn to the computation of framed open string amplitudes in both regions. In each case, we use the expressions (4.68) and (4.65) for the T -equivariant J-function, and determine recursively higher descendent insertions using topological recursions relations¹¹. At the orbifold point we have

$$\begin{aligned} \mathcal{F}_{0,1}^{\text{orb}}(t_{\text{orb}}, w, f) \\ = t_{\text{orb}} w - \frac{2f+1}{4} t_{\text{orb}}^2 w^2 + \left(\frac{1}{3} + \frac{1}{18} (-9f^2 - 9f - 2) t_{\text{orb}}^3 \right) w^3 + \dots, \end{aligned} \quad (4.91)$$

$$\begin{aligned} \mathcal{F}_{0,2}^{\text{orb}}(t_{\text{orb}}, w_1, w_2, f) \\ = \frac{9f^2 + 9f + 1}{18} t_{\text{orb}}^2 w_1 w_2 + \frac{2f+1}{2} (w_1^2 w_2 + w_1 w_2^2) + \dots, \end{aligned} \quad (4.92)$$

whereas at large radius we find, denoting again with $Q = e^t$ the exponentiated flat Kähler modulus,

$$\begin{aligned} \mathcal{F}_{0,1}^{\text{LR}}(Q, w, f) \\ = (1 - 2Q + 5Q^2) w - \frac{1}{2} \left((2f+1) (14Q^2 - 4Q + 1) \right) w^2 + \dots, \end{aligned} \quad (4.93)$$

$$\begin{aligned} \mathcal{F}_{0,2}^{\text{LR}}(Q, w_1, w_2, f) \\ = \left(-\frac{1}{2} f(f+1) + (2f^2 + 2f + 1) Q + (-7f^2 - 7f - 4) Q^2 \right) w_1 w_2 \\ + (1 + 2f) \left(\frac{1}{3} f(f+1) - (2f^2 + 2f + 1) Q + 3(3f^2 + 3f + 1) Q^2 \right) \\ \times (w_1^2 w_2 + w_2 w_1^2) + \dots. \end{aligned} \quad (4.94)$$

It is an expected, yet remarkable fact that framed open string amplitudes with non-trivial quasi-modular properties are correctly computed using our framework. For example, the B-model annulus function of local \mathbb{P}^2 is linear in the second Eisenstein series $E_2(\tau)$ [14, 18], where τ is the elliptic modulus of the genus 1 mirror curve; as a consequence, it transforms non-trivially under changes of duality frame, and the analytically continued orbifold and large radius amplitudes differ by a shift. In our language, this shift is correctly reproduced by imposing the invariance condition for the disc amplitude (4.63) and the loop insertion formula (3.7).

¹¹ To be precise, as we emphasized in Sect. 3.2.1, formulas such as (3.33) require knowledge of two-point functions with primary insertions in big quantum co-homology. However, as $QH_T^*(X)$ is generated in degree 2 in the toric case, we can express them in terms of two-point functions with one primary insertion in small quantum co-homology, which are in turn determined by the small J-function of Appendix A.

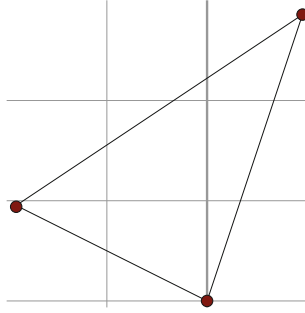


Fig. 10. The toric diagram of $[\mathbb{C}^3/\mathbb{Z}_7]$

4.4. A genus 3 example. As a last example, consider the TCY3 associated to the toric diagram in Fig. 10. The resulting singular toric variety \mathcal{X} is a \mathbb{Z}_7 -orbifold of \mathbb{C}^3 by an action with weights $(1, 1, 5)$; 1-dimensional cones in $\mathcal{F}_{\mathcal{X}}$ can be taken to be

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}. \quad (4.95)$$

The (unique) toric crepant resolution of \mathcal{X} is the local Calabi-Yau geometry of three \mathbb{F}_2 Hirzebruch surfaces glued pairwise along a \mathbb{P}^1 . The mirror geometry is given by a family \hat{X} of genus 3 projective curves over a base S with $\dim_{\mathbb{C}} S = 3$; it can be shown that the generic fiber of this family is non-hyperelliptic.

Computing open string amplitudes on \mathcal{X} would be tough using standard methods; first of all we are away from large radius, and moreover the mirror geometry makes it hard to compute basic B-model building blocks such as the Bergmann kernel. In our formalism, we can use (3.7), (3.9), (A.9) and (3.29) to address this problem, which is of the same order of computational difficulty of the $[\mathbb{C}^3/\mathbb{Z}_3]$ we treated in the last section. We denote as usual twisted classes by $\mathbf{1}_k$, $k = 0, \dots, 6$; we have

$$\deg(\mathbf{1}_k) = 2\text{age}(\mathbf{1}_k) = \begin{cases} 0 & k = 0 \\ 2 & 0 < k \leq 3 \\ 4 & 3 < k \leq 6 \end{cases}. \quad (4.96)$$

Writing t_k , $k = 1, 2, 3$ for the (small quantum co-homology) orbifold Kähler parameters, we have, for a toric brane L on one of the two legs acted on with weight 1,

$$\begin{aligned} \mathcal{F}_{0,1}^{\mathcal{X},L}(t_1, t_2, t_3, w, f) \\ = \left(t_1 - \frac{7f+8}{98}t_2t_3^2\right)w + \left(\frac{t_2}{2} - \frac{21(f+1)t_1^2}{294}\right)w^2 + \dots, \end{aligned} \quad (4.97)$$

$$\begin{aligned} \mathcal{F}_{0,2}^{\mathcal{X},L}(t_1, t_2, t_3, w_1, w_2, f) \\ = \frac{1}{98}f(f+1)t_1^2w_1w_2 + \frac{2}{147}f(f+1)t_2t_1\left(w_1^2w_2 + w_1w_2^2\right) + \dots. \end{aligned} \quad (4.98)$$

5. Summing the Instantons: Mirror Symmetry and Spectral Curves

In the last section we checked in detail that our results agree completely with the computation of A-model topological string amplitudes from string duality; a natural question to ask is whether we can recover the formalism of [3] or [13] from (3.7). In this section we begin to address this problem; our aim will be to make contact with local mirror symmetry and recover the mirror geometry by resumming A-model instantons at all orders from our localization approach. More precisely, we know that the reduction $\log y(p)$ of the holomorphic $(3, 0)$ form on the Hori-Vafa mirror curve coincides [7] with the derivative of the disc amplitude with respect to the B-model open modulus p . In our formalism, this is calculated as in (3.16): once we have computed the disc factor $D^{X,L}(d, f)$ and the twisted I -function $I^X(\mathbf{x}; \frac{1}{d}, f)$, a resummation over winding numbers in the localization formula will give us the family of spectral curves on the nose

$$\log y(p) = \sum_d d D^{X,L}(d, f) I^X\left(\frac{1}{d}, z, f\right). \quad (5.1)$$

We will now see how to recover the mirror Hori-Vafa differentials for the examples of the previous section.

5.1. The framed vertex. Let us start from the case of the framed vertex; this type of computation was considered from a different point of view¹² in [5, 13, 19, 71]. We have from (3.9) that

$$\log y(p) = \sum_{d=1}^{\infty} \frac{\Gamma(fd + d)}{d! \Gamma(df + 1)} p^d. \quad (5.2)$$

We can resum (5.2) in hypergeometric form as

$$\begin{aligned} \log y(p) &= \begin{cases} -\log(1 - p) & \text{for } f = 0 \\ p {}_fF_{f+1}\left(1, 1, \frac{f+2}{f+1}, \dots, \frac{2f+1}{f+1}; \frac{f+1}{f}, \dots, \frac{2f-1}{f}, 2, 2; \frac{(f+1)^{f+1}p}{ff}\right) & \text{for } f > 0 \end{cases} \end{aligned} \quad (5.3)$$

In order to compare with the topological vertex result, we make the $GL(2, \mathbb{Z})$ reflection $\log y(p) \rightarrow -\log y(p)$, and we redefine $p \rightarrow -p$, which amounts to a trivial shift of the bare open modulus. The resulting sign transformation at each order in d is an ubiquitous fact in open string mirror symmetry, and it should possibly correspond to the ambiguity in choosing a canonical orientation of the open string moduli space on the A-model side. We have for the redefined exponentiated variable $y(p)$ that [5]

$$\begin{aligned} y(p) &= \begin{cases} p + 1 & \text{for } f = 0 \\ \frac{f-1}{f} {}_fF_{f-1}\left(-\frac{1}{f+1}, \frac{1}{f+1}, \dots, \frac{f-1}{f+1}; \frac{1}{f}, \dots, \frac{f-1}{f}; \frac{-(f+1)^{f+1}p}{ff}\right) + \frac{1}{f} & \text{for } f > 0 \end{cases} \end{aligned} \quad (5.4)$$

¹² Remarkably, in [19] the backreaction of the toric brane to Kähler gravity in this case was found to have a formulation in terms of the A-model on non-nef local curves.

The case of negative f can be recovered from (5.4) by using the duality $f \leftrightarrow -f - 1$, which is manifest in (4.1). The latter is a hypergeometric root (see e.g. [19]) of the trinomial equation

$$xy^{-f} - y + 1 = 0 \quad (5.5)$$

which is one form of the spectral curve of the framed vertex, upon identifying the torus weight f with the Chern-Simons framing as $f \rightarrow -f$.

5.2. The resolved conifold. In this case, using (4.40) and (4.44), the localization formula reads

$$\log y(p) = \sum_{d=1}^{\infty} {}_2F_1(-d, d; 1; e^t) \frac{p^d}{d}. \quad (5.6)$$

For each fixed integer d , the hypergeometric function on the right-hand side is a Jacobi polynomial in e^t [60],

$${}_2F_1(-d, d; 1; e^t) = P_n^{0, -1}(1 - 2e^t). \quad (5.7)$$

Using the formula for the generating function of Jacobi polynomials

$$\sum_{n=1}^{\infty} P_n^{\alpha, \beta}(x) z^n = \frac{2^{\alpha+\beta}}{R(R-t+1)^{\alpha}(R+t+1)^{\beta}} - 1, \quad R = \sqrt{1 - 2xz + z^2}, \quad (5.8)$$

we obtain

$$p \partial_p \log y(p) = \frac{p+1 - \sqrt{(4e^t - 2)p + p^2 + 1}}{2\sqrt{(4e^t - 2)p + p^2 + 1}}, \quad (5.9)$$

and integrating once

$$\log y(p) = -\log \left(\frac{1}{2} \left(\sqrt{4pe^t + (1-p)^2} - p + 1 \right) \right). \quad (5.10)$$

Up to trivial re-definitions $y \rightarrow \frac{1}{y}$, $p \rightarrow x = -p$, this is just the planar resolvent of the Chern-Simons matrix model, and we obtain the family of mirror curves of the resolved conifold in the hyperelliptic, framing one form

$$1 + xy + y + e^t xy^2 = 0. \quad (5.11)$$

5.3. Local surfaces. The case of toric Calabi-Yau threefolds whose mirror curve has genus greater than zero presents no extra difficulty (see for example [17]). A lengthy, but straightforward general method to compare the sum over instantons (5.1) with the mirror geometry is to exploit the fact [18] that the derivatives of the Hori-Vafa differential with respect to the B-model closed moduli are algebraic functions of both the open and the closed moduli. This allows to compare the B-model and A-model instanton expansion of $\partial_p \log y(p)$ in a completely explicit way. The leftover ambiguity is fixed by a computation at the large radius point, which reduces to the type of sums of Sect. 5.1.

We refer the reader to [17] for the (rather lengthy) details of this computation for the case of a Hirzebruch surface \mathbb{F}_2 , and mention here the result for $K_{\mathbb{P}^2}$ with an outer brane at zero framing. Here we find from (4.68) and (4.84) that

$$\begin{aligned} \sum_d dw^d D^{X,L}(d, 0) I^X \left(z, \frac{1}{d}, 0 \right) &= \sum_{d,n} \frac{w^d z^n \Gamma(d(-f-1)+1)(d-1)!}{n!^2 \Gamma(d-3n+1) \Gamma(d(-f-1)+n+1)} \Big|_{f=0} \\ &= \log \left(\frac{1}{2} \left(1 - w - \sqrt{1 - 2w + w^2 + 4w^3 z} \right) \right) \\ &= \log y(w), \end{aligned} \quad (5.12)$$

which yields the mirror family of elliptic curves of local \mathbb{P}^2 [53],

$$y^2 + yw + y + zw^3 = 0. \quad (5.13)$$

6. Conclusions and Outlook

Our formalism opens several new lines of investigation. We mention here a few ideas for future work.

- The most pressing question is whether our approach could open the way for a fully rigorous proof of the BKMP proposal [13]. In recent work, starting from [40], it was advocated that one way to do this would be to establish the Eynard-Orantin recursion in the form of cut and join equations. Until now, this strategy has proved to be fruitful only in the case of genus zero spectral curves; yet, the line of reasoning of Sect. 5 lends itself to the study of higher genus spectral curves and higher order amplitudes, such as the Bergmann kernel. An enticing possibility would be to investigate the case of the orbifold topological vertex along the lines of [40], and then to give further substance to the arguments of Sect. 3.1.4 that extend the analysis to all chambers of the secondary fan. A second aspect to understand in our language is the global structure of topological amplitudes; while we checked that our formalism embeds automatically the non-trivial quasi-modular shift of the propagator under a change of S -duality frame, it would be nice to have a further understanding of its origin from the A-model side, including, in view of (1.1), the gravitational sector.
- From a physical point of view, it would be desirable to understand microscopically the duality (1.1), and perhaps to exhibit a worldsheet proof along the lines of [76]. The possibility of an interpretation of (1.1) from a space-time point of view is also an attracting one; on the mirror side, this should find a clear place into the formalism [12]. There is plenty to understand here already for the simple example of the resolved conifold. A further aspect to clarify in relation to (1.1) is the backreaction

of toric branes to A-model gravity; while (1.1) gives a complete answer to this problem in terms of gravitational descendants, a more satisfactory answer might perhaps be given in a geometric way as in [19]. Moreover, the deformation by descendants should have a natural interpretation whenever a dual matrix model picture is available; for example, in the case of the resolved conifold it is tempting to conjecture that the deformation (4.40), (4.44) should take the form of adding higher Casimirs in the dual sum over 2D-partitions [37].

- While our formalism is, by construction, particularly efficient in adding “holes” to the worldsheet by repeated application of the brane insertion operator $\mathcal{L}(w, f)$, it is much harder to compute $F_{g,h}^{X,L}$ at higher genus. As we mentioned, a complete and mathematically rigorous solution is given by Givental’s quantization formalism; a first line of action would be to find a place for a systematic implementation of Givental’s formula, and to see how this type of quantization relates to the mirror symmetry picture of [2]. This is the aspect we are currently devoting more attention to, and we hope to report on this in the near future.
- Speaking of quantization, the duality with closed T -equivariant Gromov-Witten invariants might bring into play new ingredients: the right hand side of (1.1) should also be described through the type of integrable structures that arise in the study of Gromov-Witten invariants from the vantage of Symplectic Field Theory [36]. In this context, *quantum dispersionless* integrable systems, as opposed to the *classical dispersionful* hierarchies of Sect. 3.2, appear in the description of the full-descendent theory; as quantum integrable systems have received much attention recently in the study of higher genus corrections to Seiberg-Witten theory [74], a possible relationship between the two theories begs for further understanding.
- A point which is completely absent in our formalism is the issue of Nekrasov’s “refinement”, as we do not have an A-model moduli space interpretation for that in terms of holomorphic worldsheet instantons. While it seems natural to conceive that (1.1) should still hold true in the Ω -background, with the same brane insertion operator of the $\beta = 1$ case, it is at the moment unclear how to implement the deformation in the dual, full-descendent closed string side.
- Whenever a connection to *known* integrable hierarchies is available, as for the resolved conifold [15], this would yield in principle a non-perturbative completion of the open topological string: integrable hierarchies arising from KP/Toda reduction as in [15] are naturally well-defined for finite g_s , (corresponding, *e.g.* in the Toda case, to finite lattice spacing), and the analogy with matrix models makes it natural to think that the master formula (1.1) should hold true non-perturbatively in g_s . Moreover, the presence of a Lax formalism would provide both a new way to derive the mirror geometries (see [16]), as well as a novel method to deform them by introducing string loop corrections.
- Finally, in the simplest example of a topological string related to gauge theory [72], integrable flows have a natural interpretation as deforming the ultraviolet Lagrangian by turning on descendants of single-trace chiral operators. It would be intriguing to generalize this statement in our context, and to analyze it in particular for the type of gauge theories that the topological string geometrically engineers.

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A. Toric Geometry and I-Functions

A.1. The projective case. We will review here the main statements of [24, 25, 47] about the I-functions for toric orbifolds. To start with, let X be a projective smooth toric variety with $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = k$, and write Z_X for its Stanley-Reisner ideal. Write X as the holomorphic quotient $X = (\mathbb{C}^n \setminus Z)/(\mathbb{C}^*)^k$; the weights of the torus action can be encoded in an integral $k \times n$ matrix $M = (m_{ij})$. Let $\{C_1, \dots, C_k\}$ be a basis of $H_2(X, \mathbb{Z})$ given by fundamental classes of compact holomorphic curves in X associated to the rows of M , $\{r_1, \dots, r_k\}$ with $r_i \in H^{1,1}(X)$ their duals in co-homology, and $\{D_1, \dots, D_n\}$ the divisors given by $z_i = 0$, where z_k is the k^{th} homogeneous co-ordinate of $\mathbb{C}^n = (z_1, \dots, z_n)$. We consider furthermore a $T \simeq (\mathbb{C}^*)^r$ multiplicative action on z_i ; we write $\mathbb{C}[\lambda_1, \dots, \lambda_r]$ for the coefficient algebra of $H_T(X)$, and write p_i for the lift of the class r_i to T -equivariant co-homology. Consider now the equivariant classes

$$u_j = \sum_{i=1}^n m_{ij} p_i - \lambda_i \quad (\text{A.1})$$

which are the Poincaré duals of the T -invariant co-ordinate hyperplanes D_j , associated to a 1-dimensional cone of the secondary fan of X ; by construction we have $\int_{C_i} u_j = m_{ij}$. The (small) I-function of X [47] is defined in terms of the toric data as the co-homology valued formal power series

$$I_X(y_1, \dots, y_k, z) = e^{\ln y_0/z + z \sum_{i=1}^k p_k \ln y_k/z} \sum_{d \in \mathbb{Z}_+^k} \prod_{j=1}^n \frac{\prod_{m=-\infty}^0 (u_j + mz)}{\prod_{m=-\infty}^{\sum_i m_{ij} d_i} (u_j + mz)} \prod_{l=1}^k y_l^{d_l} \quad (\text{A.2})$$

Let now $J_X(t_1, \dots, t_k, z)$ be the T -equivariant J-function of X ,

$$J_X(t_1, \dots, t_k; z) := z + \sum_l t_l p_l + \sum_{n=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} \left\langle \mathbf{t}, \dots, \mathbf{t}, \frac{p^l}{z - \psi} \right\rangle_{0, n+1, \beta}^{X_T} p_l, \quad (\text{A.3})$$

restricted to small quantum co-homology. Then we have the following

Theorem A.1 (Givental). *Suppose $c_1^T(X) \geq 0$. Then $J_X = I_X$ up to a (weighted homogeneous) change of variables*

$$\ln y_0 \rightarrow t_0 = \ln y_0 + z f_0(y) + h(y), \quad (\text{A.4})$$

$$\ln y_l \rightarrow t_l = \ln y_l + f_l(y), \quad l = 1, \dots, k, \quad (\text{A.5})$$

where $f_l(y)$, $h(y)$ are analytic functions in y .

In other words, when X is projective the J-function (2.16) is entirely specified by the toric data defining the I-function, up to the change of variables (A.4), (A.5). The latter in turn is uniquely determined by comparing the asymptotic expansions at large z of J_X and I_X .

A.2. The twisted case. When X is non-compact, and in particular a toric Calabi-Yau threefold, (A.2) continues to hold. The proof of this result will appear in [23], including the case of toric orbifolds. We will state here two specializations of [23], which apply to the cases we treated in Sect. 4, and whose proof has already appeared in the literature.

As a first example, let $X \rightarrow Y$ be the total space of a concave line bundle on a projective semi-positive toric variety Y , and let $T' \simeq (\mathbb{C}^*)^n$ be a torus action on Y as in the previous section. We take a $T \simeq (\mathbb{C}^*)^{n+1} \curvearrowright X$ to cover the T' -action on $Y \hookrightarrow X$ and rotate the fibers by complex multiplication; we write $\mathbb{C}[\lambda_1, \dots, \lambda_n, \lambda_{n+1}] = \mathbb{C}[\lambda_1, \dots, \lambda_n][\lambda_{n+1}]$ for the coefficient algebra of $H_T^\bullet(X)$, denoting the equivariant parameter associated to the torus action along the fibers by λ_{n+1} . Let moreover ρ be the first Chern class of X , and define the following *hypergeometric modification*:

$$M_X(d) := \prod_{b: \langle \rho, d \rangle < b \leq 0} (\lambda_{n+1} + \rho + bz). \quad (\text{A.6})$$

for d in the semigroup inside $H_2(X, \mathbb{Z})$ generated over \mathbb{Z}_+^k by the curve classes C_1, \dots, C_k . Then we have the following [24, 25].

Theorem A.2 *In terms of the J-function of Y (A.3), the I-function of X reads*

$$I_X(t_1, \dots, t_k, z) := z e^{\sum_l p_l \ln y_l / z} \left(1 + \sum_d e^{t_1 d_1} \dots e^{t_k d_k} M_X(d) \left\langle \frac{p^l}{z(z - \psi)} \right\rangle_{0,1,d}^Y p_l \right). \quad (\text{A.7})$$

A.3. Orbifolds. The second special case we need is when $X = [\mathbb{C}^3 / \mathbb{Z}_n]$; the determination of the twisted I-function of X builds once more on the CCIT-twisting procedure [24], this type applied to the Gromov-Witten theory of BG [58]. Notations here are as in Sect. 3.1.2.

Theorem A.3 *Let X be the total space of the direct sum of line bundles $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$ over $B\mathbb{Z}_n$. Let e_i be the integer such that \mathcal{E}_i is given by the character $[k] \mapsto \exp(\frac{2\pi e_i k \sqrt{-1}}{n})$ of \mathbb{Z}_n and that $0 \leq e_i < n$. For $\mathbf{l} = (l_1, \dots, l_n)$, set*

$$a_i(\mathbf{l}) = \sum_{j=1}^n l_j \left\langle \frac{(j-1)e_i}{n} \right\rangle, \quad i = 1, 2, 3. \quad (\text{A.8})$$

Then the small I-function of X reads

$$I_X(x_1, \dots, x_n, z) := \sum_{l_1, \dots, l_n} \prod_i \frac{x_i^{l_i}}{l_i! z^{l_i}} \prod_{j=1}^3 \prod_{m_j=0}^{[a_i(\mathbf{l})]-1} (\lambda_j - (\langle a_i(\mathbf{l}) \rangle + m_j z)) \mathbf{1}_{(\sum_i l_i(i-1)/n)}. \quad (\text{A.9})$$

When restricted to small quantum co-homology, the I-function (A.9) has the large z -expansion

$$I_X(x_1, \dots, x_n, z) = F(\mathbf{x})z\mathbf{1}_0 + G(\mathbf{x}) + \mathcal{O}\left(\frac{1}{z}\right), \quad (\text{A.10})$$

and the J-function is then

$$J_X(t_1, \dots, t_n, z) = \frac{I_X(x_1(t), \dots, x_n(t), z)}{F(x_1(t), \dots, x_n(t))}, \quad (\text{A.11})$$

where $t_i(\mathbf{x}) = G_i(\mathbf{x})/F(\mathbf{x})$.

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